# Psychometrika


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VOLUME ONE SEPTEMBER 1936 NUMBER THREE

PSYCHOMETRIKA—VOL. 1, NO. 3
SEPTEMBER, 1936

# BOUNDARY CONDITIONS IN THE COMMON-FACTOR-SPACE, IN THE FACTORIAL ANALYSIS OF ABILITY

*by*

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The author arrives at a simple rule for ascertaining when a matrix of correlations, with communalities reducing it to minimum rank, cannot be analyzed into factors such that every column of loadings has at least as many zeros as the number of common factors, as required by Thurstone. A more exact but arithmetically tedious rule is also deduced from Ridley Thompson's boundary conditions, and a correction is made to the latter.

## *I. Object of the present paper.*

In his latest method of factorial analysis, which is so clearly and completely described in *The Vectors of Mind,* Professor Thurstone finds those "communalities" which, when inserted in the diagonal cells of a sufficiently large* matrix of correlations, reduce its rank to the lowest possible number. He shows that the tests which give the correlations can be expressed in terms of common factors equal in number to that minimum rank, together with specific factors whose "loadings" or coefficients are the square roots of the defects of those "communalities" from unity. Thurstone gives a beautifully simple method of arriving at one set of the loadings, in each test, of this small number of common factors. These loadings which are first obtained include negative values, and Thurstone proceeds to "rotate the axes" of the common factors until there are no negative loadings, and until a large number of them are zero. He lays it down (on pages 151 and 156) that (1) each row of the matrix of loadings must have at least one zero, that is each of the tests must lack one at least of the factors; (2) that each column must have at least as many zeros as there are factors, that is there is to be no general factor, for every factor is to be absent in several tests; and (3) that in each pair of columns there must be at least as many tests, whose loadings vanish in the one but not in the other column, as there are factors in the analysis. The object of the present paper is to enquire under what circumstances Thurstone can succeed in attaining the second of these

*See Thurstone's Table 2, page 77 of *The Vectors of Mind.*

three *desiderata*. In accordance with Thurstone's notation I shall use $n$ for the number of tests, and $r$ both for the rank of the matrix, and for the number of factors. The symbol $r$ is not to be confused with a correlation coefficient, from which it is distinguished by the absence of suffixes. Further, I shall call a factor which has loadings (other than zero) in two tests, a 2-factor; one which has loadings (other than zero) in three tests, a 3-factor; and so on up to a general factor which has loadings in every test, i.e. an $n$-factor. The question to be answered is therefore, under what circumstances can a set of $n$ tests be expressed in terms of $r$ common factors (plus of course $n$ specific or 1-factors) none of which has more than $(n—r)$ loadings (leaving at least $r$ zeros)? Note that this question applies to the common-factor-space, that is, to a space of $r$ dimensions (where $r$ is the minimum rank of the correlation matrix) which is at right angles to all the specifics.

## *II. J. Ridley Thompson's Boundary Conditions.*

Now a similar question, referring to the whole of space of which the $n$ test-vectors define a subspace, has already been partly answered, and my task is therefore that of adapting that answer to the limitations of the common-factor-space. In the year 1919 I asked myself what values the three correlations of three tests must have if it were to be impossible to express these tests without a general factor running through all three of them, and I noted that the answer was, approximately, that a general factor was essential if the correlations of the three tests averaged more than 0.5, or those of $n$ tests averaged more than $(n—2)/(n—1)$. I suggested the problem as a piece of research to Mr. J. Ridley Thompson, now of Sheffield University, and he gave an answer, in determinantal form, for three tests.* Later, in a series of papers, he gradually generalized the theorem, his final form (which is in fact not quite correct) being the following statement.†

*See appendix by J. Ridley Thompson to my paper "The Proof or Disproof of the Existence of General Ability" *Brit. Journ. of Psychol.* 1919, IX, 335, and see also page 142 of *The Essentials of Mental Measurement* by Brown and Thomson.

†"The General Expression for Boundary Conditions and the Limits of Correlation", by J. Ridley Thompson, *Proc. Roy. Soc. Edinburgh*, 1928-9, Vol. XLIX, pp. 65-71. In passing it may be pointed out that in lines 12 to 18 of his page 69, Ridley Thompson came very near anticipating Thurstone's fundamental proposition on rank and factors.

If $\Delta$ be the determinant

$$\Delta = (-1)^n \begin{vmatrix} k & r_{12} & r_{13} & \cdot & r_{1n} \\ r_{12} & k & r_{23} & \cdot & r_{2n} \\ r_{13} & r_{23} & k & \cdot & r_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{1n} & r_{2n} & r_{3n} & \cdot & k \end{vmatrix} \cdot \cdot \cdot \quad (1)$$

then as $k$ is given the values $-1$, $-2$, $-3 \cdots$ and so on, (always the same value, be it noted, along the whole diagonal) the sign of $\Delta$ will change from minus to plus at the point where the higher order factors are unnecessary, as shown in this table:—

*Table I.*

| $k$ | If $\Delta$ is minus | If $\Delta$ is plus |
|---|---|---|
| $-1$ | 3-factors or higher factors are necessary. | 3-factors (and higher factors) are unnecessary. |
| $-2$ | 4-factors or higher factors are necesary. | 4-factors (and higher factors) are unnecessary. |
| . . | . . . . . | . . . . . |
| $-(n-2)$ | $n$-factors are necessary. | $n$-factors are unnecessary. |

The probable error of $\Delta$ has been given by Dr. T. P. Black* to whom also the determinantal form of the final theorem is due, for curiously enough, though J. Ridley Thompson gave his first result in that form, he did not do so in the general case.

I may point out here that Ridley Thompson's is really a theorem concerning the positions of the latent roots of the correlation matrix $R$, that is the roots of the characteristic equation

$$| R - \lambda I | = 0$$

and that Ridley Thompson's diagonal quantity $k$ is equivalent to $1 - \lambda$. When I realized this, towards the end of 1935, I thought at first that Ridley Thompson's theorem must imply that a correlational matrix cannot have more than one latent root greater than 2, (corresponding to his $k = -1$), for if, for example, 4-factors and higher factors are unnecessary, then the sign of $\Delta$ should not again change.

*Mental Measurement: The Probable Error of Some Boundary Conditions in Diagnosing the Presence of Group and General Factors," *Proc. Roy. Soc. Edinburgh* 1928-9, Vol. XLIX, pp. 72-77.

But Dr. Aitken, to whom I mentioned this, replied by constructing a correlation matrix which did possess two roots greater than 2, and Mr. Walter Ledermann has since made several such for me. Clearly therefore Ridley Thompson's theorem as he stated it is incomplete. The sign of $\Delta$ may change several times, and it is only the last of these changes to which his theorem refers and then only as a necessary, and not a sufficient condition. This may be expressed by saying that if $\lambda$ is the *largest* latent root, then $(s+1)$ factors or higher factors are necessary (where $s+1$ is the integer immediately above $\lambda$). Mr. Walter Ledermann is at present inquiring into the whole connection of factor-extent and latent-roots.*

This difficulty in the application of Ridley Thompson's theorem—the difficulty, that is, of being certain that when the sign changes to plus it will remain plus — makes the simple approximate method now to be described more useful, for this difficulty does not arise there.

### *III. Approximate Solution of the same Problem.*

The approximate solution of the general factor problem which I gave above, namely that when the average of the correlations exceeds $(n-2)/(n-1)$ a general-factor is essential, can be readily extended to cover all the boundary conditions for $n$ variables. It is simply that when the average of the correlations exceeds $(s-1)/(n-1)$, $s$ being a positive integer, an $(s+1)$-factor or a higher factor is essential. This expression is derived by considering the case where there are *only* $s$-factors, including *all possible* $s$-factors, and where all loadings are equal. In that case all the correlations are alike, and are equal to $(s-1)/(n-1)$, and this is the highest average amount of correlation that can be produced without going higher than $s$-factors. The method of using this rule therefore is to calculate the average of all the correlation coefficients, and to note where this average value fits in, in the series of fractions

$$\frac{1}{n-1}, \frac{2}{n-1}, \frac{3}{n-1}, \cdots \frac{n-2}{n-1}.$$

Thus if $n$ is 21 (say), and the average correlation coefficient is .632, this value comes between 12/20ths and 13/20ths

$$\frac{12}{20} < .632 < \frac{13}{20}$$

and this shows that at least one 14-factor or higher factor is necessary. Both Ridley Thompson's exact theorem, and my approximate substi-

*See his paper immediately following this article.

tute for it, when applied to the measured correlations, are concerned with all possible analyses of the tests, and are not confined to the common-factor-space.

### *IV. Application of Boundary Conditions to Correlations Corrected for Communality.*

In order to apply these theorems to the common-factor-space it is necessary to "correct" the correlations first for communality. Each test-vector can be divided into two components, one (its specific) along an axis at right angles to the common-factor-space, the other *in* the common-factor-space. It is the latter with which we are to deal, and the cosines of the angles between them are given by

$$r'_{ij} = \frac{r_{ij}}{h_i\, h_j} \cdots \tag{2}$$

where $r_{ij}$ is the correlation coefficient, and $h_i^2$ and $h_j^2$ are the two communalities. These latter quantities $r'_{ij}$ therefore it is into whose matrix the diagonal elements $k = -1, -2$, etc. have to be inserted in applying the boundary conditions to the common-factor-space only:—

$$\Delta' = (-1)^n \begin{vmatrix} k & r'_{12} & r'_{13} & \cdot & r'_{1n} \\ r'_{12} & k & r'_{23} & \cdot & r'_{2n} \\ r'_{13} & r'_{23} & k & \cdot & r'_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r'_{1n} & r'_{2n} & r'_{3n} & \cdot & k \end{vmatrix} \cdots \tag{3}$$

It is however unnecessary to calculate the quantities $r'_{ij}$. Instead we note that

$$\Delta' = \frac{1}{h_1^2\, h_2^2\, h_3^2 \cdots h_n^2} (-1)^n \begin{vmatrix} kh_1^2 & r_{12} & r_{13} & \cdot & r_{1n} \\ r_{12} & kh_2^2 & r_{23} & \cdot & r_{2n} \\ r_{13} & r_{23} & kh_3^2 & \cdot & r_{3n} \\ r_{1n} & r_{2n} & r_{3n} & \cdot & kh_n^2 \end{vmatrix} \tag{4}$$

Since the communalities $h_i^2$ are all positive quantities, the sign of $\Delta'$ depends upon the sign of the determinant on the right. The rule therefore is to insert into the diagonal cells of the ordinary correlation matrix $k$ times the communality, and examine the sign of the resulting determinant multiplied by $(-1)^n$. The rotation desired by Thurstone, giving r zeros in each column of loadings, is impossible, if the sign

be minus for $k = -(n-r-1)$, for $(n-r+1)$-factors or higher factors will then be essential.

Similarly, the approximate method of section III can be applied, the only change being that it is the average of the *"corrected"* correlation coefficients ("corrected" for communality) which is to be used. A numerical illustration will illuminate both methods.

Before turning to the numerical illustration I should however refer to one point regarding the application of these theorems, or at least of my own, to the common-factor-space. I arrived at my theorem by the consideration that, when every $s$-factor is used, the maximum attainable correlation is $(s-1)/(n-1)$. In the common-factor-space however not every $s$-factor can be used, for the number of columns of loadings is fixed by the rank. This imposes a new condition, and will make it still more difficult to attain a high average correlation without the use of an $(s+1)$-factor. It may be expected therefore that an $(s+1)$-factor will be doubly essential if the average correlation exceeds $(s-1)/(n-1)$. Light is thrown on this by Mr. Ledermann's paper.

### *V. Arithmetical Illustration: Approximate Method.*

Consider the following matrix of correlation coefficients

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | *.89* | .44 | .47 | .40 | .48 | .56 | .40 | .30 | .35 |
| 2 | .44 | *.25* | .24 | .15 | .18 | .21 | .32 | .24 | .28 |
| 3 | .47 | .24 | *.25* | .20 | .24 | .28 | .24 | .18 | .21 |
| 4 | .40 | .15 | .20 | *.89* | .46 | .83 | .32 | .56 | .24 |
| 5 | .48 | .18 | .24 | .46 | *.40* | .54 | .08 | .14 | .06 |
| 6 | .56 | .21 | .28 | .83 | .54 | *.85* | .24 | .42 | .18 |
| 7 | .40 | .32 | .24 | .32 | .08 | .24 | *.80* | .76 | .68 |
| 8 | .30 | .24 | .18 | .56 | .14 | .42 | .76 | *.85* | .63 |
| 9 | .35 | .28 | .21 | .24 | .06 | .18 | .68 | .63 | *.58* |

which can be reduced to rank 3 by the communalities shown. Let us inquire, first of all by the approximate method, whether after analysis into three common factors, there is the possibility that the axes can be rotated in the common-factor-space until there are three zeros in

every column. The 36 correlations corrected for communality have to be averaged, which can most readily be done by deleting the communalities in the matrix, dividing each column by the root of that column's communality, adding the rows, and dividing each total by the root of that row's communality. The grand total must then be divided by 72. Without troubling to keep many decimals (for the method is in any case only approximate) we obtain .585 for the average of the corrected correlations. In the fraction $(s—1)/(n—1)$ the denominator equals 8, and .585 comes between 4/8ths and 5/8ths. The numerator $s—1$ is therefore 4, and so $s$ is equal to 5, that is 6-factors or higher factors are essential. This leaves the possibility open that three zeros may occur in each column. The actual analysis, after rotation, is:—

FACTOR LOADINGS

| Test | I | II | III | Communality |
|---|---|---|---|---|
| 1 | .8 | .5 | 0 | .89 |
| 2 | .3 | .4 | 0 | .25 |
| 3 | .4 | .3 | 0 | .25 |
| 4 | .5 | 0 | .8 | .89 |
| 5 | .6 | 0 | .2 | .40 |
| 6 | .7 | 0 | .6 | .85 |
| 7 | 0 | .8 | .4 | .80 |
| 8 | 0 | .6 | .7 | .85 |
| 9 | 0 | .7 | .3 | .58 |

If however the correlations in the last row and column of the matrix, instead of being as shown, had been

.75 .43 .41 .49 .36 .53 .68 .63

the communality of the ninth test, to give rank 3, would have been .83. If for this modified matrix we find the average of the correlations corrected for communality we obtain .633, which comes between 5/8ths and 6/8ths so that $s—1 = 5$, and $s$ is 6, that is a 7-factor or higher factor is essential. In at least one column therefore the zero-loadings must be less than three. The actual analysis and rotation gives:—

FACTOR LOADINGS

| Test | I | II | III | Communality |
|---|---|---|---|---|
| 1. | .8 | .5 | 0 | .89 |
| 2. | .3 | .4 | 0 | .25 |
| 3. | .4 | .3 | 0 | .25 |
| 4. | .5 | 0 | .8 | .89 |
| 5. | .6 | 0 | .2 | .40 |
| 6. | .7 | 0 | .6 | .85 |
| 7. | 0 | .8 | .4 | .80 |
| 8. | 0 | .6 | .7 | .85 |
| 9. | .5 | .7 | .3 | .83 |

It is impossible to obtain three zeros in every column with this battery.

*VI. Arithmetical Illustration: Exact Method.*

If three or more zeros are to be possible in each column of loadings, the determinant of equation (4) multiplied by $(-1)^n$, must be positive for $k = -(n-r-1) = -(9-3-1) = -5$. The arithmetical task therefore is to evaluate the determinant with —5 times the communalities in the diagonal. This is not a formidable task if Aitken's method of pivotal condensation* is used, but the comparative ease of the approximate method makes it unlikely that this exact method will be employed except for theoretical inquiries. Mr. Ian Wishart has performed the calculations for me and finds.

$$\text{for } k = -5, \quad \Delta' = +\ 1705.5 \div h_1^2 \cdots h_9^2$$

$$\text{for } k = -4, \quad \Delta' = -\ 816.0 \div h_1^2 \cdots h_9^2 \ .$$

A 6-factor or higher factor is therefore essential, as we found with the approximate method. When the last row and column of the matrix are altered as before to the higher correlations of the alternative test 9, we find that the value of $\Delta'$ changes, for $k = -5$, to $-115.5 \div h_1^2 \cdots h_9^2$, so that, with these correlations, at least one 7-factor is nec-

*"Note on the Computation of Determinants", by A. C. Aitken, *Transactions of the Faculty of Actuaries* (Edinburgh) 1931, Vol. XIII, pp. 12-15. See also my paper "Some Points of Mathematical Technique in the Factorial Analysis of Ability", *Journ. of Educ. Psychol.* January 1936, pp. 45-6.

essary, and Thurstone's requirement of three zeros per column certainly cannot be obtained.

## *VII. Summary.*

The paper inquires into the *necessary* conditions under which a matrix of correlations of minimal rank $r$ can be analyzed into $r$ column factors with at least $r$ zeros in every column of loadings. An approximate but sufficiently exact answer is given, which involves no heavy calculations, namely that if the average of the correlations "corrected" for communality exceeds $(s-1)/(n-1)$, where $n$ is the number of tests and $s$ is a positive integer, then $s$-factors are insufficient, an $s$-factor being one which has non-vanishing loadings in $s$ of the tests, leaving $(n-s)$ zeros. A more stringent condition is also arrived at, by applying Ridley Thompson's "boundary conditions" to the matrix with correlation coefficients corrected for communality and with various negative integers in the diagonal.

In passing it is pointed out that Ridley Thompson's theorem is not completely stated by him, inasmuch as the sign of the determinant occurring in it may change several times in special cases. The last change is that to which his remarks apply, that is, it is the *largest* latent root of the matrix which is significant in this connection.

PSYCHOMETRIKA—VOL. 1, NO. 3
SEPTEMBER, 1936

# SOME MATHEMATICAL REMARKS CONCERNING BOUNDARY CONDITIONS IN THE FACTORIAL ANALYSIS OF ABILITY.

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This paper is a mathematical supplement to the preceding paper by Professor Godfrey H. Thomson. It gives rigorous proofs of theorrems enunciated by him and by Dr. J. Ridley Thompson, and extends them. Its basic theorem is that if a matrix of correlations is to be factorized without the aid of higher factors than $s$-factors (with $n$—$s$ zero loadings), then the largest latent root of the matrix must not exceed the sum of the $s$ largest communalities on the diagonal.

## I.

This paper is intended as a mathematical supplement to the preceding article* on "Boundary Conditions in the Common-factor-space, in the Factorial Analysis of Ability" by Professor G. H. Thomson. It was pointed out there ("*B*" II), that some difficulties arise when the boundary conditions first established by Mr. J. Ridley Thompson (see "*B*", footnotes where references may be found) are applied in their original form. It seemed therefore desirable to revise Mr. J. R. Thompson's work and to restate his results in a more mathematical and rigorous form. I would like to express my thanks to Professor G. H. Thomson for suggesting to me this problem as a piece of mathematical research.

## II.

Both Mr. J. R. Thompson's boundary condition and Professor Thomson's criterion which is a less stringent but for practical purposes more useful substitute for it, can be based on the following theorem† concerning the largest latent root of certain matrices. *If A is a real matrix with n rows and m columns,*

$$A = \begin{vmatrix} a_{11}\, a_{12} \cdots a_{1m} \\ a_{21}\, a_{22} \cdots a_{2m} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n1}\, a_{n2} \cdots a_{nn} \end{vmatrix}$$

*Henceforth referred to as "*B*".

†Proved by the writer in a paper read at a meeting of the Edinburgh Mathematical Society on June 6th, 1936 and to be published later.

*which has not more than s non-vanishing entries in each column and whose rows have norms*

$$h^2_1, h^2_2, \cdots, h^2_n \ , \quad \text{i.e.}$$

$$\sum_{k=1}^{m} a^2_{ik} = h^2_i \qquad (i = 1, 2, \cdots, n) \ ,$$

*then the largest latent root of the (non-negative, symmetric) matrix*

$$R = A\,A'$$

*does not exceed the quantity,*

$$\sigma = h^2_{a_1} + h^2_{a_2} + \cdots + h^2_{a_s} \ , \tag{1}$$

*where $h^2_{a_1}, h^2_{a_2}, \cdots, h^2_{a_s}$ are the s largest norms.*

If, in particular, the rows of $A$ be *normalized,* i.e.

$$h^2_1 = h^2_2 = \cdots = h^2_n = 1 \ ,$$

then $h^2_{a_1} + h^2_{a_2} + \cdots + h^2_{a_s} = s$ is an upper limit for the latent roots of $R$.

III.

The application of this theorem to Factorial Analysis is obvious. We identify $R$ with the matrix of correlations and $A$ with the matrix of loadings. If $A$ contains the common-factors as well as the specific factors (see "B" I), then the rows of $A$ will be normalized and it follows that all latent roots of $R$ must be less than or equal to $s$, the maximum number of non-vanishing entries in each of the columns of $A$. On the other hand, when we confine ourselves to the common-factor-space, the number of columns in $A$ will be $r$ and the norms of the rows of $A$ are the "communalities"

$$h^2_1, h^2_2, \cdots, h^2_n \ ,$$

which are less than 1. Hence the upper limit $\sigma$ for the latent roots of $R$ is in this case less than $s$. If $\omega$ be the largest latent root of $R$, i.e. the largest root of the determinantal equation

$$\Delta(\lambda) = (-)^n \mid R - \lambda I \mid = 0 \ , \tag{2}$$

then

$$\Delta(\lambda) > 0 \ ,$$

when

$$\lambda > \omega \ .$$

Ridley Thompson's boundary condition may therefore be enunciated as follows (*c.f.* "B" II and IV):

*If a matrix* R *of correlations is to be factorized without the aid of higher factors than* s*-factors* ($0 \leqq s < n$), *then the latent roots of* R *must not exceed the quantity*

$$\sigma = h^2_{a_1} + h^2_{a_2} + \cdots + h^2_{a_s} ,$$

*where* $h^2_{a_1}, h^2_{a_2}, \cdots, h^2_{a_s}$ *are the* $s$ *largest diagonal entries of* $R$.

or:

*If a matrix* R *of correlations is to be factorized without the aid of higher factors than* s*-factors, then the determinant* $\Delta(\lambda)$ *should be positive when* $\lambda > 0$.

Stating the theorem in yet another form we may say (*c.f.* "B"):

*If* $\mu$ *is an integer and if the largest latent root of* R *exceeds* $\mu$, *or if* $\Delta(\mu) < 0$, *then* $\mu$*-factors are certainly insufficient to account for the given correlations and factors with more than* $\mu$ *non-vanishing loadings are needed.*

I would like to point out that this boundary condition—and still more so the following criteria—are *necessarily* fulfilled when $s$-factors are sufficient to explain the observed correlations. But it has not been proved that, conversely, a factorial matrix can actually be found which contains only $s$-factors or lower factors, provided these conditions are satisfied.

## IV.

We shall now deduce Professor Thomson's criterion ("B" III) from Ridley Thompson's boundary condition. Let $\omega$, as before, be the largest latent root of $R$; as is shown in the theory of quadratic forms,* $\omega$ may be defined as the maximum of the quadratic form

$$x\,R\,x' = \sum_{i,k} r_{ik}\,x_i\,x_k \qquad (i, k = 1, 2, \cdots, n) \ , \tag{3}$$

when $x$ ranges over all unit (row)-vectors

$$x = [x_1, x_2, \cdots, x_n] \ , \quad \text{i.e. when}$$

$$\sum_{i=1}^{n} x_i^2 = 1 \ . \tag{4}$$

Hence for every such vector we have

$$\sum r_{ik}\,x_i\,x_k \leqq \omega \ . \tag{5}$$

*See e.g. H. W. Turnbull and A. C. Aitken, *An Introduction to the Theory of Canonical Matrices*, p. 170. (Glasgow, 1932).

In particular, if we take the unit vector

$$x_0 = \frac{1}{\sqrt{n}}\,[1, 1, \cdots, 1] \ , \qquad \text{i.e. } x_i = \frac{1}{\sqrt{n}} \ .$$

we obtain by (5):

$$\frac{1}{n}\sum_{i,k} r_{ik} \leqq \omega \ . \tag{6}$$

We now introduce the *average correlation*

$$\bar{r} = \sum_{i<k} r_{ik} \Big/ \binom{n}{2} = \frac{2\sum_{i<k} r_{ik}}{n(n-1)} \tag{7}$$

and the *average communality*

$$\bar{h}^2 = \frac{h^2_1 + h^2_2 + \cdots + h^2_n}{n} \ . \tag{8}$$

Thus we can write the left hand member of (6):

$$\frac{1}{n}\sum_{i,k} r_{ik} = \frac{1}{n}\left[(h^2_1 + h^2_2 + \cdots + h^2_n) + \sum_{i \neq k} r_{ik}\right]$$

$$= \frac{1}{n}\left[(h^2_1 + h^2_2 + \cdots + h^2_n) + 2\sum_{i<k} r_{ik}\right] ,$$

since

$$r_{ik} = r_{ki} \ ;$$

and substituting (7) and (8):

$$\frac{1}{n}\sum_{i,k} r_{ik} = \bar{h}^2 + (n-1)\bar{r} \leqq \omega \ .$$

Hence

$$\bar{r} \leqq \frac{\omega - \bar{h}^2}{n-1} \ . \tag{9}$$

If we consider the total factor-space, we have to replace all communalities by unity and hence also

$$\bar{h}^2 = 1 \ .$$

Further, by §II, the number $s$ is in this case an upper limit for $\omega$. We therefore get by (9):

$$\bar{r} \leqq \frac{s-1}{n-1} \tag{10}$$

(See "B" ··· ). Consequently, if $\bar{r}$ exceeds $\frac{s-1}{n-1}$, an $(s+1)$-factor or higher factor is essential for the analysis of the observed correlations.

## V.

*A theorem by G. Frobenius.* In order to apply Ridley Thompson's boundary condition it is necessary to find an integer which approximates to the largest root of the equation

$$\Delta(\lambda) = (-)^n \mid R - \lambda I \mid = \mid \lambda I - R \mid = 0 \ . \qquad (11)$$

Let $\omega$ be the largest root of equation (11). If $\mu > \omega$, then $\Delta(\mu) > 0$, for $\Delta(\lambda)$ does not change its sign after $\lambda$ has passed $\omega$, and $\Delta(\lambda) > 0$ for sufficiently large values of $\lambda$. On the other hand, if $\Delta(\mu) < 0$, then $\mu$ is certainly less than $\omega$.

In most practical cases, the correlations i.e. the elements of the matrix $R$ are *positive* numbers and general theorems which deal with those matrices become applicable.

Professor O. Perron has proved* that the largest latent root of a matrix with positive elements is always a simple root so that the sign of $\Delta(\lambda)$ changes as $\lambda$ passes from values less than $\omega$ to values greater than $\omega$.

In many cases, especially when the correlation coefficients do not greatly differ, the following theorem by G. Frobenius† will be found useful.

Let $R = [r_{ik}]$ be the matrix of correlations in which the diagonal entries may either be the communalities or all equal to unity. Denote by $r_1, r_2, \cdots, r_n$ the sums of the elements in the first, second, ··· $n$th column (row). The largest latent root then lies between $r_m$ and $r_n$:

$$r_m \leqq \omega \leqq r_n \ , \qquad (12)$$

where $r_m$ is the least and $r_n$ the greatest of those sums.

To illustrate this theorem we shall use the same example that was given in "B", §V. In order to apply Ridley Thompson's boundary conditions we have to estimate the largest latent root of the matrix ("B" §IV, (2) )

$$R^* = [\frac{r_{ij}}{h_i h_j}] \ , \qquad (13)$$

*See O. Perron, *Algebra Bd. II*, p. 37 (Berlin 1933).

†G. Frobenius: Über Matrizen aus positiven Elementen. *S. B. preuss. Akad. Wiss* 1908, pp. 471-476.

where the values of $r_{ij}\,(i \neq j)$ and $h^2_i = r_{ii}$ are given in the table on p. 160. The characteristic determinant of $R^*$ then becomes:

$$\Delta^*(\lambda) = (-)^9 \begin{vmatrix} 1-\lambda & .94 & 1.00 & .46 & .81 & .65 & .48 & .35 & .49 \\ .94 & 1-\lambda & .96 & .32 & .57 & .46 & .72 & .52 & .74 \\ 1.00 & .96 & 1-\lambda & .43 & .76 & .61 & .54 & .39 & .55 \\ .46 & .32 & .43 & 1-\lambda & .78 & .96 & .38 & .65 & .34 \\ .81 & .57 & .76 & .78 & 1-\lambda & .93 & .15 & .24 & .12 \\ .65 & .46 & .61 & .96 & .93 & 1-\lambda & .29 & .50 & .26 \\ .48 & .72 & .54 & .38 & .15 & .29 & 1-\lambda & .92 & .91 \\ .35 & .52 & .39 & .65 & .24 & .50 & .92 & 1-\lambda & .89 \\ .49 & .74 & .55 & .34 & .12 & .26 & .91 & .89 & 1-\lambda \end{vmatrix} .$$

For the sums of the elements in the columns of $R^*$ we obtain:

$$r^*_1 = 6.18 \;;\; r^*_2 = 6.23 \;;\; r^*_3 = 6.24 \;;\; r^*_4 = 5.32 \;;\; r^*_5 = 5.36 \;;$$
$$r^*_6 = 5.66 \;;\; r^*_7 = 5.39 \;;\; r^*_8 = 5.44 \;;\; r^*_9 = 5.30 \;.$$

By Frobenius's theorem the largest latent root therefore lies between 5.30 and 6.24. This is in accordance with the fact ("B" VI) that $\Delta^*(\lambda)$ is positive for $\lambda = 6\,(k = -5)$ and negative for $\lambda = 5\,(k = -4)$.

## VI.

Special interest is attached to the case in which

$$m = r \text{ and } s = n - r \;,$$

where $r$ is the rank of $R$ and also the number of factors i.e. the number of columns in the factorial matrix $A$. The discussion of this case is the main object of Professor Thomson's paper. If we put $s = n - r$ in (1), (9) or (10), we obtain conditions which must be fulfilled when the matrix of loadings is to contain not more than $n - r$ non-vanishing entries, or at least $r$ zero entries, in each column.

It is perhaps of some theoretical interest to have further necessary conditions which must be satisfied in this particular case. In chapter VI of his book "The Vectors of Mind", Professor Thurstone discusses the problem of finding a suitable set of coordinate axes with respect to which the matrix of loadings assumes a simple shape. The most general factorial matrix can be written in the form

$$B = A\,L \;,$$

where $A$ is a special factorial matrix and $L$ any square matrix of or-

der $r$ whose determinant does not vanish and whose columns are normalized (see Thurstone, *loc. cit.* p. 154 where our $B$, $A$, $L$, are denoted by $V$, $F$, $G$). It is desired to determine $L$ in such a way that $B$ has at least $r$ zeros in each of its columns. What conditions must $A$ or $AA' = R$ fulfill if that transformation is to be possible?
Let

$$A = \begin{vmatrix} a_{11}\, a_{12} \cdots a_{1r} \\ a_{21}\, a_{22} \cdots a_{2r} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n1}\, a_{n2} \cdots a_{nr} \end{vmatrix} = \begin{vmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{vmatrix} , \tag{14}$$

where

$$a_v = [a_{v1}, a_{v2}, \cdots, a_{vr}] \quad (z = 1, 2, \cdots, r)$$

is a row-vector with $r$ elements. Similarly, let

$$L = [l_1, l_2, \cdots, l_r] ,$$

where

$$l_z = \{l_{1z}, l_{2z}, \cdots, l_{rz}\} \quad (\varrho = 1, 2, \cdots, r)$$

is a column-vector with $r$ elements. We can then write

$$B = \begin{vmatrix} b_{11}\, b_{12} \cdots b_{1r} \\ b_{21}\, b_{22} \cdots b_{2r} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ b_{n1}\, b_{n2} \cdots b_{nr} \end{vmatrix} = \begin{vmatrix} (a_1\, l_1)\, (a_1\, l_2) \cdots (a_1\, l_r) \\ (a_2\, l_1)\, (a_2\, l_2) \cdots (a_2\, l_r) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (a_n\, l_1)\, (a_n\, l_2) \cdots (a_n\, l_r) \end{vmatrix} , \tag{15}$$

where

$$b_{ik} = (a_i\, l_k) = \sum_{j=1}^{r} a_{ij}\, l_{jk} \quad (i = 1, 2, \cdots, n\ ;\ k = 1, 2, \cdots, r) \tag{16}$$

is the inner product of the vectors $a_i$ and $l_k$. Suppose now that the factorial matrix $B$ is of the required form. Then $r$ entries in the first column of the right hand member of (15) should vanish. Let these be the entries:

$$(a_{a_1}\, l_1) = (a_{a_2}\, l_1) = \cdots = (a_{a_r}\, l_1) = 0 \, (a_1 < a_2 < \cdots < a_r) \ . \tag{17}$$

We can condense (14) in one matrix equation:

$$A_a\, l_1 = 0 \ , \tag{18}$$

where

$$A_\alpha = \begin{vmatrix} a_{\alpha_1} \\ a_{\alpha_2} \\ \cdot \\ a_{\alpha_r} \end{vmatrix} = \begin{vmatrix} a_{\alpha_1 1} a_{\alpha_1 2} \cdots a_{\alpha_1 r} \\ a_{\alpha_2 1} a_{\alpha_2 2} \cdots a_{\alpha_2 r} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{\alpha_r 1} a_{\alpha_r 2} \cdots a_{\alpha_r r} \end{vmatrix} . \tag{19}$$

This is one of the $\binom{n}{r}$ minors of order $r$ that can be formed from the given matrix $A$. Since $l_1 \neq 0$, it follows that the determinant of $A$ vanishes:

$$| A_\alpha | = 0 . \tag{20}$$

In analogy to (18) we get for the second column of $B$:

$$A_\beta l_2 = 0 , \tag{21}$$

where $A_\beta$ is likewise a minor of $A$ of order $r$, not necessarily distinct from $A_\alpha$. Again, we must have

$$| A_\beta | = 0 .$$

By repeating this argument for all columns of $L$ we obtain $r$ minors

$$A_\alpha, A_\beta \cdots\cdots, A_\rho \tag{22}$$

each being of zero determinant. We shall first assume that all these minors are formally distinct.

In order to express our result in a concise form we say (following Sylvester) that a square matrix is of *"nullity"* $\nu$ if its order exceeds its rank by $r$; hence we have

$$\textit{nullity} = \textit{order} \text{ minus } \textit{rank} . \tag{23}$$

Every matrix of non-vanishing determinant is of nullity zero, irrespective of its order. If the determinant of a matrix vanishes, its nullity is necessarily positive ($\geqq 1$). The nullity is, of course, never negative. Each of the minors (22) is of nullity $\geqq 1$, while the remaining $\binom{n}{r} - r$ minors are of nullity $\geqq 0$. We may therefore say that the sum of the nullities of all minors of order $r$ is at least $r$. This statement still holds when some of the minors (19) are identical. For suppose that

$$A_\alpha = A_\beta ,$$

i.e. that $A_\alpha$ and $A_\beta$ are obtained by selecting the same rows from $A$. By (18) and (21) we have then

$$A_\alpha l_1 = 0 \; ; \quad A_\alpha l_2 = 0 .$$

Since $L$ is of non-zero determinant, $l_1$ and $l_2$ are linearly independent, and it follows that the set of homogeneous equations

$$A_\alpha x = 0$$

has (at least) two linearly independent solutions. According to well known theorems* concerning linear equations this implies that $A_\alpha$ cannot be of rank higher than $r - 2$, or that the nullity of $A_\alpha$ is at least 2. Thus the sum of the nullities of $A_\alpha$ and $A_\beta$ is again $\geqq 2$ and our statement obviously holds for any number of repetitions in the set (22).

It is possible to apply our test directly to the given matrix $R$ without the aid of a provisional factorial matrix $A$. For by definition we have

$$A\,A' = R = [r_{ik}] \ ,$$

where $A$ is defined in (14). By actually performing the matrix multiplication we obtain

$$r_{ik} = (a_i\, a_k) = \sum_{j=1}^{r} a_{ij}\, a_{kj} \ , \quad (i, k = 1, 2, \cdots, n) \ . \tag{24}$$

Similarly, it is readily proved that

$$A_\alpha\, A'_\alpha = \begin{vmatrix} (a_{\alpha_1} a_{\alpha_1}) & (a_{\alpha_1} a_{\alpha_2}) & \cdots & (a_{\alpha_1} a_{\alpha_r}) \\ (a_{\alpha_2} a_{\alpha_1}) & (a_{\alpha_2} a_{\alpha_2}) & \cdots & (a_{\alpha_2} a_{\alpha_r}) \\ & & & \\ (a_{\alpha_r} a_{\alpha_1}) & (a_{\alpha_r} a_{\alpha_2}) & \cdots & (a_{\alpha_r} a_{\alpha_r}) \end{vmatrix} = R_\alpha, \text{ say,}$$

where $A_\alpha$ is defined in (19). Substituting (24), we see that $R_\alpha$ is a principal minor of order $r$ formed from $R$ by selecting the rows and columns with suffixes $\alpha_1, \alpha_2, \cdots \alpha_r$, thus

$$R_\alpha = \begin{vmatrix} r_{\alpha_1\alpha_1} & r_{\alpha_1\alpha_2} & \cdots & r_{\alpha_1\alpha_r} \\ r_{\alpha_2\alpha_1} & r_{\alpha_2\alpha_2} & \cdots & r_{\alpha_2\alpha_r} \\ \cdot & \cdot & \cdot & \cdot \\ r_{\alpha_r\alpha_1} & r_{\alpha_r\alpha_2} & \cdots & r_{\alpha_r\alpha_r} \end{vmatrix} \ .$$

Now, it is known (see Thurstone, *loc. cit.* p. 72) that $R_\alpha$ and $A_\alpha$ are of the same rank and hence also of the same nullity. We have therefore proved the following theorem:

*If it is to be possible to factorize a correlational matrix $R$ of rank $r$*

*See e.g. H. W. Turnbull, *The Theory of Determinants, Matrices and Invariants*, London and Glasgow, (1929), p. 75.

*in such a way that the matrix of loadings contains at least r zeros in each of its columns, then the sum of the nullities of the* $\binom{n}{r}$ *r-rowed principal minors of R must at least be equal to r.*

Consequently, if the sum of those nullities does not reach the value $r$, the factorisation required will be impossible. Since the rank of $R$ is $r$, some of the principal $r$-minors do certainly not vanish (cf. Thurstone, *loc. cit.* p. 91). But it may very well occur and, indeed, will in general be the case that *none* of those minors vanishes, or that they are all of nullity zero. It will then be impossible to find a matrix $L$ with the desired properties.

A little consideration will show that the last criterion is more stringent than the other two given above which are, however, not restricted to the case $s = n - r$. For it may happen that the first two criteria still leave open the question whether a transforming matrix $L$ can be found while this possibility is excluded by the last criterion.

PSYCHOMETRIKA—VOL. 1, NO. 3
SEPTEMBER, 1936

# THE FACTORIAL ISOLATION OF PRIMARY ABILITIES

L. L. THURSTONE

This is an experimental study of the isolation, by factor methods, of primary abilities from a battery of tests given to 240 students. The range and nature of the fifty-six tests is briefly described. Tentative interpretations of the twelve orthogonal primary factors are given.

This is a brief summary of an experimental study of human abilities. The plan of the experiment was to cover a wide range of the paper-pencil tests that are in current use for appraising intelligence. The test battery was assembled so as to include, abstraction, space, visualizing, form, number, reasoning, and verbal material. If primary abilities are present in these tests, the factorial methods should be adequate for their isolation since the battery was fairly extensive, including fifty-six tests.

The complete battery required fifteen hours of testing time. The subjects were 240 student volunteers at the University of Chicago. All the subjects who were included in the factorial analysis completed the whole battery. Only 9% of the volunteers failed to finish all the tests. Most of the subjects were freshmen and sophomores at the University of Chicago, but the experimental group was not strictly confined to them. A few older subjects were allowed to participate. Each of the volunteer subjects who completed the whole battery was given a report on his performance on each of the tests in comparison with the rest of the group. In making these preliminary reports the tests were grouped as well as could be done in the present state of knowledge without factorial analysis by the constituent elements.

The nature of the tests will be described here with a brief mention of each form. Most of the tests were designed in three parts, namely, the instructions with examples, a fore-exercise, and the test proper. The fore-exercise was not necessary for a few of the tests, such as spelling and arithmetic, in which the nature of the task was readily understood from general instructions. The number in parentheses after each test is the code number by which the test is identified throughout the analysis.

*List of Tests in the Battery*

*Matched Proverbs* (4) for similar meaning,

*Matched Quotations* (5),

These two tests were designed by the writer at Carnegie Institute of Technology and they have been used by Professor Thorndike in the CAVD tests.

*Verbal Classification* (6) was intended to be a verbal parallel to Spearman's

*Figure Classification* (8),

*Word Grouping* (7) in which the subject selects the odd word in a group of five words,

*Controlled Association* (9) in which the subject is asked to write as many words as he can recall whose meaning is similar to a given word,

*Inventive Opposites* (10) and *Inventive Synonyms* (16) in which the subject is asked to recall and to write the response words,

*Completion* (11) is similar to the test with the same name in the Psychological Examination of the American Council on Education.

*Disarranged Words* (12) in which the subject rearranges pied letters of a word,

*First and Last Letter* (13) in which the subject writes as many words as he can recall that begin and terminate with specified letters,

*Disarranged Sentences* (14) was adapted from the Army Alpha test,

*Anagrams* (15) is similar to other tests by the same name,

*Block Counting* (17) is adapted from a test by MacQuarrie,

*Cubes* (18) is adapted from a test by Brigham,

*Lozenges A* (19), *Lozenges B* (22), and *Flags* (20) were designed by the writer some years ago as tests of space thinking,

*Form Board* (21) is the Minnesota Paper Form Board which was here used with permission of the authors,

*Surface Development* (23) is a new test of space thinking in which the subject is asked to make comparisons between a flat cardboard or sheet metal form and the sketch of the object into which the sheet can be folded,

*Punched Holes* (24) is a development of the well known test,

*Mechanical Movements* (25) is an assembly of pictures of mechanical movements in which the subject is asked to specify the movement of certain parts when that of other parts is given,

*Identical Forms* (26) in which the subject must find one of five designs which is identical with a given design,

*Pursuit* (27) and *Copying* (28) are adapted from two tests by MacQuarrie,

*Areas* (29) is adapted from a test by Brigham,

*Number Code* (30) is a new number test which involves the use of the Mayan number code,

*Addition* (31), *Subtraction* (32), *Multiplication* (33), *Division* (34) are simple speed tests of arithmetical computation,

*Tabular Completion* (35), to fill in the missing entries of a table by inspecting the other entries and the column headings of the table,

*Estimating* (36) is a modified form of a test of the same name which was used in the 1924 Psychological Examination of the American Council of Education,

*Number Series* (37) is a well known form of test,

*Numerical Judgment* (38), to select one of four given numerical answers as the correct one without carrying out the complete computations. In most of the problems the answer can be given by noting the order of magnitude of the numbers so that detailed computation is superfluous,

*Arithmetical Reasoning* (39) is a test in current use,

*Reasoning* (40) contains a series of syllogisms and *False Premises* (42) is a syllogistic test in which all of the terms are meaningful words which, however, do not make meaningful premises or conclusions. The subject must ascertain whether the conclusion follows from the premises although both are apparently nonsensical statements,

*Verbal Analogies* (41) is similar to the well known form,

*Code Words* (43) is an extension of a reasoning test by Godfrey Thomson,

*Pattern Analogies* (44) is similar to one of the tests in the Psychological Examination of the American Council on Education,

*Syllogisms* (45) is characterized by monotony of content,

*Rote Learning I* (46), to memorize paired associates. The stimuli are words and the responses are two digit numbers,

*Rote Learning III* (47) consists of memorizing paired associates with names as stimuli and initials as responses,

*Rote Learning II* (48) consists of memorizing paired associates with two digit numbers as stimuli and other two digit numbers as responses,

*Word Recognition* (49) is a recognition memory test for individual words with specially selected lists of distractors,

*Figure Recognition* (50) is a recognition memory test for simple geometrical figures with similiar distractors,

*Picture Recall* (51) calls for memory of the detail of a picture,

*Theme* (52) is a short theme in which the subjects were asked to describe an acquaintance. The themes were rated by English instructors,

*Hands* (53), to indicate for each of a large number of pictures of hands whether it is a right or a left hand,

*Rhythm* (54) is a test with groups of four lines of poetry. The subject is asked to select the one line whose rhythm is different from the other three,

*Sound Grouping* (55), to indicate which one of each group of five words sounds different from the other four,

*Spelling* (56),

*Grammar* (57),

*Two Vocabulary tests* (58 and 60) are well known forms. In the Grammar test the subject was asked to correct faulty sentences by altering only one word,

*Free Writing* (59) was merely the word count in the theme.

Most of the frequency distributions of raw scores were clearly unimodal and approximately symmetric. In the present study it is assumed that whatever the primary abilities may be they are normally distributed in the experimental population. If this assumption is true then any linear combination of the primary abilities must also be normally distributed. With this assumption a deviation from normality in the distribution of raw scores is regarded as an arbitrary effect of the degree of difficulty of the items. This assumption seems to be the safest that we can make about these psychological functions. It is certainly less arbitrary than to assume that the distribution of raw scores is itself significant since we know that such distributions can be altered at will by changing the proportions of items of differ-

ent degrees of difficulty. The ideal procedure would therefore seem to be to normalize all of the distributions before entering them into correlation tables. The most complete procedure would be to compute Pearson product-moment coefficients. The assumption of the normality of the underlying primary distributions and the resulting normality of their linear combinations fits exactly the assumption that underlies the tetrachoric correlation coefficient. It is, in fact, an estimate of the Pearson product-moment coefficient for pairs of normally distributed variates. Tetrachoric correlation coefficients were used in this investigation. They were computed by dividing each of the variates at the median, at which the determination is more accurate than when the division is made near either tail of the distribution. The tetrachoric correlation coefficients were determined by facilitating tables* by which four determinations of the coefficient can be made in a few minutes after the appropriate four-fold tables have been assembled from a Hollerith tabulator. This procedure sacrifices accuracy in that the probable error of a tetrachoric correlation is larger than the probable error of a Pearson product moment coefficient. However, this procedure is superior to that of computing Pearson product-moment coefficients for the raw scores in case some of the distributions deviate noticeably from normality. This work is largely a matter of computational routine.

The correlational matrix was of order 57x57 and it was factored to twelve dimensions by the centroid method.†

By far the largest part of the labor is in the rotation of a twelve dimensional configuration so as to bring it into a positive orthogonal manifold. This work has been done by several analytical methods and by graphical methods of determining orthogonal transformations. The graphical methods have been in use for several years and a complete example will probably be published in a later issue of *Psychometrika.*

The factorial analysis of a correlational matrix consists essentially of two parts. The first problem is to factor the correlational matrix. In the present case the factorial matrix is of order 57x12. The factoring can be done routinely by the centroid method or by any other method which does not disturb the minimum rank of the correlational matrix with unknown diagonals. The second major part of the problem is to rotate the configuration into a unique position with reference to the coordinate axes so that the axes shall be scientifically

*Chesire, Saffir, Thurstone, Computing Diagrams for the Tetrachoric Correlation Coefficient, (*The University of Chicago Bookstore*, 1933).

†L. L. Thurstone, The Vectors of Mind (Chicago: *The University of Chicago Press*, 1935), Chapter III.

meaningful. The unique solution is attained for psychological abilities when the factorial matrix has been so rotated that the number of zero entries is maximized and so that the non-vanishing entries are positive. An arbitrary factorial matrix cannot be rotated so that a large proportion of the entries become vanishingly small and so that the non-vanishing entries are positive. When this can be done the test battery has revealed an underlying order. It is the object of factorial analysis to discover that order.

When the factorial matrix of the present problem was rotated into a positive orthogonal manifold, the entries in the columns of the factorial matrix were studied to obtain a psychological interpretation of each factor. Seven of the primary abilities seem to have fairly clear psychological meaning. Two others can be given a tentative psychological interpretation. One of the factors seems to be an error factor and two of the factors do not seem to be sufficiently conspicuous in the present battery to allow clear interpretation.

Perhaps the most conspicuous of the factors is the one which has strong factor loadings on all of the tests (30-39) inclusive except (36). The highest factor loadings are on (30), (31), (32), and (33). A glance at the corresponding tests reveals that this is the whole number group. All of the tests without exception that are high on this factor are number tests. The simplest number tests have the highest factor loadings in this factor. The more complex number tests, namely, (35-39) inclusive have relatively lower factor loadings on this factor. We seem justified in denoting this factor $N$ and if we should name it we should naturally call it a *number factor*. At this point it may be well to call attention to the fact that the naming of a factor is always necessarily limited to the available tests which represent it. It may happen that this factor which we have denoted $N$ or *number* might transcend those functions which we ordinarily refer to as numerical. If we should discover some other tests which also have high loadings on this factor and which are non-numerical in character, then our psychological interpretation of this factor would have to be extended to some more fundamental function which would still include numerical operations as good examples but which would not necessarily be limited to numerical operations.

The next conspicuous factor has a heavy loading in the following tests: (18), (20), (22), (23), and (27) and it has marked factor loadings also in (8), (17), (19), (21), (45), and (53). If we introspect about the mental activities that are involved in doing these tests, it seems clear that the common element is visualizing. Flat space and solid space are both included in this factor. There seems justification, therefore, for denoting this factor $V$ or *visualizing*.

Another factor which seems to be rather easily interpreted psychologically has large factor loadings in tests (46-48) inclusive and appreciable factor loadings in (49-51) inclusive. These are the six memory tests in the present battery. It is of some interest to note that all six of these tests combine into a single memory factor. Although two of them involve numbers they do not appear in the number factor. However, work with specially designed batteries is indicated to ascertain whether memory can be split into several distinct memory factors. This factor has been denoted *M* and it might be named *memory* in terms of the tests by which it is identified.

Another factor has large loadings for tests (12), (15), (56), and (57) and appreciable factor loadings in (10), (13), (16), (49), (55), and (60). In all of these tests without exception the subject deals with single words. None of these tests is characterized by sustained verbal thinking. It has therefore been denoted $W_1$ in view of the fact that another verbal factor is also found to be distinct from this word factor. It might be named *word facility* or *word fluency.*

Another verbal factor has high factor loadings in tests (4), (5), (7), (9), (10), (16), (41), (57) and it has appreciable loadings in (13), (14), (40), (42), (55), (58), and (60). All of these tests are characterized by the fact that they involve word meanings or word relations. It seems to be more logical in character than the factor whose chief examples are Disarranged Words and Anagrams. Since it is distinctly verbal in character it has been denoted $W_2$ and it might be named *verbal relations* to distinguish it from the less logical factor that is implied in $W_1$ It would be interesting to investigate these two factors experimentally in relation to some forms of aphasia. It is likely that experimental studies of abnormalities may enable us to clarify the psychological nature of these factors more definitely than experiments on normal subjects.

A more restricted factor has high loadings in (6), (7), (26), and (51), the highest loading being for Identical Forms. Considerable study of the tests that have high and appreciable loadings on this factor seems to indicate the psychological interpretation that it involves facility in a type of perceptual discrimination. A special test battery is now being arranged to study this factor in further detail. It seems to be the factor by which some individuals can scan a field of perceptual detail to identify readily what they are looking for. The factor is definitely perceptual in character and it is not limited to sensory acuity. It has been denoted *P* and it has been tentatively called *perceptual speed* although this name is subject to revision with further experimental study.

Perhaps one of the most interesting factors is that which is markedly present in tests (8), (29), (35), and (37). The latter is a Number Series test and (8) is Spearman's Figure Classification. In all of these tests the subject is asked to find some rule or principle by which given perceptual material is unified. Because of this inductive characteristic the factor has been denoted *I* and it has been tentatively named *induction*. It calls for further experimental study with test batteries specially designed to feature induction.

The appearance of induction as one of the primaries makes one immediately wonder whether deduction can be found in one of the other factors. This seems to be indicated in the fact that one of the factors has conspicuous loadings in Reasoning and in False Premises and appreciable loadings in Mechanical Movements and in Spearman's Figure Classification test. All four of these tests have in common the characteristic that certain premises are given and that one of several given inferences must be selected as correct. This factor has therefore been denoted *D* and it has been tentatively named *deduction*. If Reasoning appears with at least these two factors it would seem to follow that some individuals might be superior in deductive reasoning without being superior in induction and vice versa. This might be psychologically defensible.

One of the remaining factors is heavily loaded with Arithmetical Reasoning, Arithmetical Judgment, Vocabulary tests, Mechanical Movements, Estimating, Reasoning, and False Premises. This factor is more difficult to name but it may be characterized by some elements of precision.

All of these primary factors are uncorrelated in the experimental population.

A four hour test program involving these primaries is now being assembled for use during Freshman Week at the University of Chicago. We expect to rate each of the students separately on each of these primary abilities.

Before we shall be certain of the psychological characterization of these primary factors it will be necessary to study experimentally specially designed test batteries in which each of these primaries is featured in turn. Such studies will eventually refine our knowledge of the exact nature of each of the primary mental abilities.

PSYCHOMETRIKA—VOL. 1, NO. 3
SEPTEMBER, 1936

# THE METHOD OF MINIMUM VARIATION FOR THE COMBINATION OF CRITERIA*

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The problem of weighting separate criterion variates is solved by minimizing the differences among the standard scores of the individual upon the various measures. The method is compared with Horst's procedure of maximizing the inter-individual differences. An application is made to personnel data.

How to combine several criterion variates, estimates of the degree of excellence of performance of a task, into a single composite has had a number of solutions — all of which are necessarily concerned with the determination of the proper weights to assign the criterion variates. These include weighting the several variates proportionally to their importances as judged by "experts," weighting proportional to the reliability of the variate, weighting proportional to the average of the correlations of the one variate with the remaining variates, weighting the criterion variates so as to obtain a maximum correlation with several independent or predicter variates, extracting by factor analysis those parts of the criterion variates which may be ascribed to the same factor, or weighting the criterion variates so that the discrimination between all possible pairs of individuals in the sample will be as great as possible.

The list of methods is not all inclusive, but is presented to suggest some of the ways in which the problem has been met.

Any method for combining several criterion variates into a single composite assumes that they are measures of the same thing or various aspects of the same thing. From this assumption it follows that one should expect the differences among the variate values for any individual, expressed as standard scores, to be small. This forms the basis for the method of minimum variation for combining several criterion variates into a single composite.

*This technique was developed in connection with a study of the prediction of success of salespersons, which is being carried on by the Occupational Research Program of the United States Employment Service.

Suppose that there are given $n$ estimates of competence in the performance of a task for each of $N$ individuals, each estimate being expressed as a standard score $z_i$ whose weight $a_i$ is to be determined.

Let it be assumed that the best linear combination of these $n$ criterion variables is that one which makes the sum of squares of the $n(n-1)/2$ differences of the standard scores for any individual a minimum.

Let,

$$F(a_i) = [(a_1z_1 - a_2z_2)^2 + (a_1z_1 - a_3z_3)^2 + \cdots + (a_{n-1}z_{n-1} - a_nz_n)^2] \tag{1}$$

where the $a$'s are connected by the relation,

$$H(a_i) = a_1^2 + a_2^2 + \cdots + a_n^2 = k^2 \ , \quad k \neq 0 \tag{2}$$

The function to be minimized is then

$$S = F(a_i) + \beta H(a_i) \tag{3}$$

where $\beta$ is a Lagrange multiplier.

The first partial derivatives of $S$ with respect to the independent variables $a_1, a_2, \cdots, a_n$ are set equal to zero:

$$\left.\begin{aligned} \frac{\delta S}{\delta a_1} &= (n-1+\beta)a_1z_1^2 - a_2z_1z_2 - a_3z_1z_3 - \cdots - a_nz_1z_n = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\delta S}{\delta a_n} &= (n-1+\beta)a_nz_n^2 - a_1z_1z_n - a_2z_2z_n - \cdots - a_{n-1}z_{n-1}z_n = 0 \end{aligned}\right\} \tag{4}$$

Summing these derivatives for the $N$ individuals, $n$ normal equations are obtained. *E.g.*, the first equation of (4) becomes

$$\Sigma \frac{\delta S}{\delta a_1} = (n-1+\beta)a_1\Sigma z_1^2 - a_2\Sigma z_1z_2 - \cdots - a_n\Sigma z_1z_n = 0 \tag{5}$$

Since $\Sigma z_i^2 = N$ and $\Sigma z_iz_j = Nr_{ij}$, the normal equations may be written

$$\left.\begin{aligned} \frac{1}{N}\Sigma \frac{\delta S}{\delta a_1} &= (n-1+\beta)a_1 - r_{12}a_2 - \cdots - r_{1(n-1)}a_{n-1} - r_{1n}a_n = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{N}\Sigma \frac{\delta S}{\delta a_n} &= -r_{1n}a_1 - r_{2n}a_2 - \cdots - r_{(n-1)n}a_{n-1} + (n-1+\beta)a_n = 0 \end{aligned}\right\} \tag{6}$$

In order that these $n$ equations hold simultaneously the determinant of the $a$'s must equal zero.

$$\Delta(a) = \begin{vmatrix} n-1+\beta & -r_{12} & \cdots & -r_{1(n-1)} & -r_{1n} \\ -r_{12} & n-1+\beta & \cdots & -r_{2(n-1)} & -r_{2n} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ -r_{1n} & -r_{2n} & \cdots & -r_{(n-1)n} & n-1+\beta \end{vmatrix} = 0 \qquad (7)$$

$\Delta(a)$ vanishes for each of the $n$ real negative roots of

$$C_0\beta^n + C_1\beta^{n-1} + C_2\beta^{n-2} + \cdots + C_n\beta^0 = 0 \ , \qquad (8)$$

the expansion of $\Delta(a)$.

The coefficients of $\beta$ in (8) are obtained by evaluating the principal minors of $\Delta(a)$: $C_0 = 1$, $C_1$ equals the sum of the first order principal minors, $C_2$ the sum of the second order principal minors, $\cdots$, $C_n$ the sum of the $n$th order principal minor.

If $C_n$ equals zero, one value of $\beta$ is zero, showing that the normal equations were consistent and could be solved immediately for the weights $a_i$. If $C_n$ does not equal zero, (8) must be solved for the root nearest to zero (thus making a minimum correction to (6)). This value of $\beta$ should be substituted in (7) and the determinant proved to equal zero, before substituting it in (6).

Since there are $n$ equations with $n$ unknowns and all the constants are zero, only the ratios of the weights can be obtained. By arbitrarily giving any weight a definite value the other weights are fixed by the ratios obtained. E. g., the ratios $\frac{a_1}{a_n}, \frac{a_2}{a_n}, \cdots \frac{a_{n-1}}{a_n}$, can be computed by dividing each of the normal equations (6) by $a_n$ and solving $(n-1)$ of them. The ratios will satisfy the $n$th equation also.

The best estimate of an individual's excellence of performance, which is the linear combination of the $n$ estimates weighted so that the sums of squares of the differences among all pairs of them is a minimum, may now be written

$$t = a_1z_1 + a_2z_2 + \cdots + a_nz_n \ ,$$

or in terms of the original units, $X_1, \cdots, X_n$,

$$T = \frac{a_1}{\sigma_1}X_1 + \frac{a_2}{\sigma_2}X_2 + \cdots + \frac{a_n}{\sigma_n}X_n + (M_0 - \Sigma\frac{a_i}{\sigma_i}M_i) \ .$$

Example: The correlations among five variates: cost per cent, number of sales per day, total net sales per day, net dollars per sale, and ratio of actual credits to total net sales; taken from a department

store study of the Occupational Research Program are here shown substituted in (7).

$$\begin{vmatrix} 4+\beta & .049 & .160 & -.026 & -.098 \\ .049 & 4+\beta & -.436 & .637 & .480 \\ .160 & -.436 & 4+\beta & -.289 & .151 \\ -.026 & .637 & -.289 & 4+\beta & -.408 \\ -.098 & .480 & .151 & -.408 & 4+\beta \end{vmatrix} = 0 \qquad (9)$$

Expanded, this becomes

$$\beta^5 + 20\beta^4 + 158.8627\beta^3 + 626.2180\beta^2 + 1224.4571\beta + 949.5588 = 0 .$$

The root nearest zero is found to be $\beta = -2.9272$. Substituting this value of $\beta$ in the normal equations (6) the ratios, $\frac{a_1}{a_5}$, $\frac{a_2}{a_5}$, $\frac{a_3}{a_5}$, and $\frac{a_4}{a_5}$ are found to be .2305, —1.2132, —.3991, and .9987 respectively. If $a_5$ be arbitrarily assigned the weight of unity, the composite score for any individual in the sample is

$$t = .2305z_1 - 1.2132z_2 - .3991z_3 + .9987z_4 + z_5 .$$

Horst* developed a method for combining criterion variables so that the standard deviation of the composite scores, or the discrimination between all possible pairs of individuals in the sample, shall be a maximum. His normal equations were written,

$$\left.\begin{array}{l} (1-\psi)a_1 + r_{12}a_2 + \cdots + r_{1n}a_n = 0 , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ r_{1n}a_1 + r_{2n}a_2 + \cdots + (1-\psi)a_n = 0 . \end{array}\right\} \qquad (10)$$

where $\psi$ is the variance of the composite scores for the group.

It may be seen that if (10) be multiplied through by minus one, the equations are similar to (6), and $\psi = n+\beta$. The ratios of the weights obtained by Horst's method and the method of Minimum Variation are identical.

The above example was also worked by Horst's method. The determinant expanded was

$$-\psi^5 + 5\psi^4 - 8.8627\psi^3 + 6.7220\psi^2 - 1.9775\psi + .1107 = 0 .$$

The largest of the five real positive roots is $\psi = 2.0728$.

Testing both functions by means of the second partial derivatives, it is seen that in the method of Minimum Variation $\beta$ must be

*Horst, Paul. Obtaining a Composite Measure from a Number of Different Measures of the Same Attribute. *Psychometrika*, 1936, 1, 53-60.

$< n - 1$ for a minimum, and in Horst's method $\psi$ must be $> 1$ for a maximum. The values of $\beta$ and $\psi$ obtained in the example satisfy these inequalities.

Since the two methods result in identical sets of weights, it might be interesting to compare them on other bases. Horst's function was derived so that the composite would show a maximum differentiation among individuals. This is statistically a desirable condition. The method of Minimum Variation makes the assumption that the differences among the $n$ standard scores for an individual are a minimum. Apparently, the two approaches, with considerable difference in the logic of their derivations, are complementary.

From the computational point of view it is simpler to find the negative root nearest zero for the value of $\beta$ in (8) than to locate all the positive roots for $\psi$ in Horst's equation to be sure the largest was obtained. Horst gave an approximation method for finding the weights when $n$ is large:

$$a_1 = C\Sigma r_{i1},\ a_2 = C\Sigma r_{i_2}, \cdots, a_n = C\Sigma r_{i_n}\ .$$

The approximate weights obtained for the above example, letting $C = \dfrac{1}{\Sigma r_{i5}}$, were 1.0457, .3086, 1.6160, 1.2411, and 1, showing no agreement with the weights obtained by the exact solution.

PSYCHOMETRIKA—VOL. 1, NO. 3
SEPTEMBER, 1936

# THE CONTENT RELIABILITY OF A TEST

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The content unreliability of an essay test is the error due to the items used or the content of the test. The reader unreliability is due to variation in judgment of the persons who read and score the essay test. The content reliability of an essay test is accordingly defined as being independent of the reader reliability. Formulae are derived for the reader reliability and for the content reliability. The content reliability is found to be equal to the geometric mean of the test reliabilities computed from the scores assigned by the two readers, divided by the reader reliability.

In comparing the reliability of an objective test with that of an essay test, the interpretation of the results is complicated by the fact that there is one source of error in essay tests which does not appear in objective tests, namely, the unreliability due to variation in judgment of the reader. For the purpose of analysis, it may be said that the unreliability of an objective test is due only to the *content unreliability,* i.e., the error due to the items used or *content* of the test, while the unreliability of an essay test is due to two factors, *content unreliability* and *reader unreliability.*

Questions such as the following arise in comparing essay and objective test material. If an objective test with a reliability of .90 were changed into essay form, and could be graded with a reader reliability of .80, what is the expected reliability of the essay test? If an essay and objective test are being compared and it is found that the essay test has a reliability of .60 and a reader reliability of .80, while the reliability of the objective test is .90, can it be said that the difference in reliability is due solely to the variability of reading? A further analysis of the concept of reliability is needed in order to answer such questions.

For the purpose of analyzing and comparing the reliabilities of an essay and an objective test, it is necessary to distinguish three different types of reliability:

*total test reliability* or *observed reliability,* obtained by correlating two comparable forms of the test;

*reader reliability,* obtained by correlating the scores assigned by one reader with those assigned by another on the same material;

*content reliability*, which is the same as *test reliability* in the case of an objective test but must be determined by further analysis in case of an essay test.

The test reliability is usually defined as the correlation between two comparable forms. However, in the usual case, two comparable forms are not available so the reliability is determined by dividing the examination into halves, correlating one with the other and correcting for length by means of the Spearman-Brown formula. The reader reliability, on the other hand, is computed without the use of the Spearman-Brown formula. This fact, together with the desirability of allowing for the possibility that the error of one reader may be greater than that of another, makes it desirable to derive the content reliability, using test reliability computed by the split-half method, and correcting for length.

Let the two halves of the test be designated by the subscripts $a$ and $b$ and let the scores assigned by the two readers be designated by by $'$ and $''$. If we now regard any given test score as the sum of three quantities,

$t$, the true score,

$c$, the error due to the test content, and

$e$, the error due to the variable judgment of the reader,

we have the following definitions:

Definitions

$$x'_a = t + c_a + e'_a \ ,$$

$$x''_a = t + c_a + e''_a \ ,$$

$$x'_b = t + c_b + e'_b \ ,$$

$$x''_b = t + c_b + e''_b \ .$$

For convenience we may use $y$ to designate the test score as freed from reader error,

$$y_a = t + c_a \ ,$$

$$y_b = t + c_b \ ,$$

and designate the *total* score assigned by a reader as follows:

$$x' = x'_a + x'_b \ ,$$

$$x'' = x''_b + x''_b \ .$$

These symbols all represent *deviation* scores.

Certain assumptions made during the derivation will be set down here for completeness.

Assumptions

a) All intercorrelations between $t$, $c_a$, $c_b$, $e'_a$, $e''_a$, $e'_b$, and $e''_b$ are zero.

b) $\Sigma c^2_a = \Sigma c^2_b$. These will both be designated by $\Sigma c^2$ (without a subscript).

c) Similarly $\Sigma e'^2_a = \Sigma e'^2_b$, both designated by $\Sigma e'^2$, and

d) $\Sigma e''^2_a = \Sigma e''^2_b$, both designated by $\Sigma e''^2$.

e) $\Sigma e'^2$ is in general different from $\Sigma e''^2$.

Assumption a) states that the errors are independent of each other and of the true score. Assumptions b), c), and d) state that the error variance in one-half of the test is equal to the error variance in the other half. This is a part of what is involved in saying that the two halves are *comparable*. Assumption e) means that one reader may be much more accurate than the other.

The quantity to be solved for is the content reliability which will be designated by $r_c$. Applying the Spearman-Brown formula gives

$$r_c = \frac{2r_{y_a y_b}}{1 + r_{y_a y_b}} . \tag{1}$$

Then from the definitions we have

$$r_{y_a y_b} = \frac{\Sigma (t + c_a)(t + c_b)}{\sqrt{\Sigma (t + c_a)^2 \Sigma (t + c_b)^2}} . \tag{2}$$

Expanding and using assumptions a) and b), we get

$$r_{y_a y_b} = \frac{\Sigma t^2}{\Sigma t^2 + \Sigma c^2} . \tag{3}$$

Substituting $N$ times the variance for the sum of the deviations squared and dividing both the numerator and denominator by $N$, we get

$$r_{y_a y_b} = \frac{c^2_t}{c^2_t + c^2_c} . \tag{4}$$

Substituting (4) in (1) and simplifying, we get

$$r_c = \frac{2c^2_t}{2c^2_t + c^2_c} . \tag{5}$$

This is the content reliability. The problem is to express it in terms of the reader reliability and total test reliability.

The reader reliability ($r_r$) is given by

$$r_r = r_{x'x''} . \tag{6}$$

From the definitions it can be seen that

$$r_r = \frac{\Sigma (2t + c_a + c_b + e'_a + e'_b)(2t + c_a + c_b + e''_a + e''_b)}{\sqrt{\Sigma (2t + c_a + c_b + e'_a + e'_b)^2 \Sigma (2t + c_a + c_b + e''_a + e''_b)^2}} . \tag{7}$$

Expanding and applying assumptions a) to e), inclusive, we get

$$r_r = \frac{4\Sigma t^2 + 2\Sigma c^2}{\sqrt{(4\Sigma t^2 + 2\Sigma c^2 + 2\Sigma e'^2)(4\Sigma t^2 + 2\Sigma c^2 + 2\Sigma e''^2)}} . \tag{8}$$

Substituting $N$ times the variance for the sum of the deviations squared and dividing numerator and denominator by $2N$, we get

$$r_r = \frac{2c^2_t + c^2_c}{\sqrt{(2c^2_t + c^2_c + c^2_{e'})(2c^2_t + c^2_c + c^2_{e''})}} , \tag{9}$$

from which we have

$$2c^2_t + c^2_c = r_r \sqrt{(2c^2_t + c^2_c + c^2_{e'})(2c^2_t + c^2_c + c^2_{e''})} \tag{10}$$

The total test reliability, computed from scores assigned by reader 1, will be designated by $r_{o'}$ and is given by the formula

$$r_{o'} = \frac{2r_{x'_a x'_b}}{1 + r_{x'_a x'_b}} . \tag{11}$$

The value of $r_{x'_a x'_b}$ from the definitions is

$$r_{x'_a x'_b} = \frac{\Sigma (t + c_a + e'_a)(t + c_b + e'_b)}{\sqrt{\Sigma (t + c_a + e'_a)^2 \Sigma (t + c_b + e'_b)}} . \tag{12}$$

Expanding and applying the assumptions a) to e), inclusive, we get

$$r_{x'_a x'_b} = \frac{\Sigma t^2}{\Sigma t^2 + \Sigma c^2 + \Sigma e'^2} \tag{13}$$

Substituting $N$ times the variance for the sum of the deviations squared and dividing numerator and denominator by $N$, we have

$$r_{x'_a x'_b} = \frac{c^2_t}{c^2_t + c^2_c + c^2_{e'}} . \tag{14}$$

Substituting (14) in (11) and simplifying, we get

$$r_{o'} = \frac{2c^2_t}{2c^2_t + c^2_c + c^2_{e'}} . \tag{15}$$

Similarly, the reliability of the test as scored by reader 2 will be represented by $r_{o''}$. This is given by

$$r_{o''} = \frac{2c^2_t}{2c^2_t + c^2_c + c^2_{e''}} . \tag{16}$$

To allow for possibility that one reader may read more accurately than another it will be assumed that $c^2_{e'} \neq c^2_{e''}$ so that $r_{o'}$ and $r_{o''}$ need not be identical. From (15) and (16) we have

$$2c^2_t = \sqrt{r_{o'} r_{o''}} \sqrt{(2c^2_t + c^2_c + c^2_{o'})(2c^2_t + c^2_c + c^2_{e''})} . \tag{17}$$

Substituting (17) and (10) in (5) and simplifying give

$$r_c = \frac{\sqrt{r_{o'} r_{o''}}}{r_r} , \tag{18}$$

in which $r_c$ is the content reliability,

$r_{o'}$ is the test reliability computed from the scores of reader 1 by the split-half method corrected for length by the Spearman-Brown formula,

$r_{o''}$ is the test reliability similarly computed from the scores assigned by reader 2, and

$r_r$ is the reader reliability obtained by correlating the total test scores assigned by one reader with those assigned by another.

The resemblance between formula (18) and the correction for attenuation can easily be seen.

If the two readers are equally accurate,

$$r_{o'} = r_{o''} = r_o . \tag{19}$$

Substituting (19) in (18) gives

$$r_c = \frac{r_o}{r_r} , \tag{20}$$

in which $r_c$ is the content reliability,
$r_o$ is the test reliability, and
$r_r$ is the reader reliability.

Formula (20) will now be used to answer the illustrative questions raised at the beginning of the paper. If an objective test with a reliability of .90 were changed into an essay form which could be graded with a reader reliability of .80, how high would we expect the observed reliability of the essay test to be? Since the observed and content reliability are the same in the case of the objective test, we know that its content reliability is .90. If the content reliability of the essay test were the same (.90) and its reader reliability .80, these values may be substituted in formula (20), giving

$$.90 = \frac{r_o}{.80} ,$$

from which

$$r_o \text{ equals } .72 .$$

Therefore it can be said that the reader reliability of .80 would be expected to lower the observed reliability of the essay examination to .72.

At no point in this derivation is it assumed that the content reliability of a test is independent of a change in the form (e. g. objective to essay) of a test. The formula given enables one to correct for reader error and so find out whether or not the content reliability has been altered by a change in the form of the test.

PSYCHOMETRIKA—VOL. 1, NO. 3
SEPTEMBER, 1936

# THE FOUNDATIONS OF PSYCHOMETRY: FOUR FACTOR SYSTEMS.

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Four methods of factorizing the fundamental matrices used in factor analysis are described and illustrated. The first is represented by the techniques already developed. The second is the obverse factor technique. The third and fourth methods are variants of the first and second. The implications of each method for different schools of psychology are pointed out. The methods are complementary, not competitive.

Contents:

Introduction:
The Four Systems.
An Example.
Two Methodological Approaches.
Further Observations.
Conclusion.

## *Introduction.*

I am to draw attention to four different ways in which matrices of the kind used in factor analysis can be standardized and analyzed for factors. In the past only one of these four has been clearly distinguished, namely, that for the factor technique of Spearman, Thurstone, Kelley and others in which tests etc. are variables, and persons are the population. The second, the obverse or inversion of the first, was described recently;* it employs persons as variables instead of tests or other single attributes. Two others, of minor immediate interest and subsidiary to the first pair, will be described in the sequel. The four together, I believe, supply the foundations for a considerable part of the future science of psychometry.

In the present paper , after defining the systems and supplying an example of each, I proceed to examine the modes of thought that each can subserve. The four help us to see in perspective what, I believe, has not been perceived clearly hitherto. Two of the four are statistical statements of a relativistic standpoint, and may be of help to gestalt theorists, whilst the other two are parallel foundations for

*Stephenson, W. *The Inverted Factor Technique.* Brit. J. Psychol. XXVI, 1936, p. 344. (Other papers are referred to in the sequel).

'elemental', process,* or ability psychology. Psychologists of the former school of thought have claimed that theirs supplies the only scientific procedure.† Psychologists of the latter school draw attention to the "irrelevancies of experience" in gestalten, and hold that it is precisely characteristic of scientific methodology that it does not engulf the "total situation".§ I shall not need to take sides in this controversy. Of the four systems two subserve the former viewpoint, and two the latter; but no one of the four systems, nor either of the psychological standpoints, is sufficient or adequate alone. The future psychometry cannot hope to be complete unless it uses all four systems, each for ends which it is best fitted to serve.

*The Four Systems*

We are to be concerned with matrices of the following kind: Each is to contain persons in the row (persons $A, B, C \cdots N$) and items such as personality traits, tests, measurements or attributes in general, in the column (attributes $a, b, c, \cdots n$) .

*Persons*

| *Attributes etc.* | | $A$ | $B$ | $C$ | $D$ | .. | .. | $N$ |
|---|---|---|---|---|---|---|---|---|
| | $a$ | ${}_aX_A$ | ${}_aX_B$ | ${}_aX_C$ | .. | .. | .. | ${}_aX_N$ |
| | $b$ | ${}_bX_A$ | ${}_bX_B$ | ${}_bX_C$ | .. | .. | .. | ${}_bX_N$ |
| | $c$ | ${}_cX_A$ | ${}_cX_B$ | ${}_cX_C$ | .. | .. | .. | ${}_cX_N$ |
| | .. | .. | .. | .. | .. | .. | .. | |
| | $n$ | ${}_nX_A$ | ${}_nX_B$ | ${}_nX_C$ | .. | .. | .. | ${}_nX_N$ |

The variate of person A for item $a$, or of item $a$ for person A, is represented by ${}_aX_A$, and the same connotation is held throughout. The four systems or ways of treating such matrices are as follows:—

Variates can be entered in the matrix for single attributes or tests etc. (such as $a$), for standardization with respect to the population of persons. This is what has been done, of course, in all previous factor analysis. Data is standardized along the row, so that for each attribute or quality $a$ to $n$ respectively, the following holds:—

*Line, W. *Process-Psychology — Individual and General.* Psychol. Rev. XL. p. 256.

†Lewin, K. *Dynamic Theory of Personality,* Chap I. (McGraw-Hill) 1935.

§Thurstone, L. L. *The Vectors of Mind.* p. 46. (Chicago Univ. Press )1935.

$$\Sigma\, {}_a x^2 = \Sigma\, {}_b x^2 = \cdots = \Sigma\, {}_n x^2 = 1 \ . \qquad (1)$$

(where the summation is from $x_A$ to $x_N$ in each case). Rows are correlated, and analyzed by way of factor theorems such as those of Spearman, Thurstone, Hotelling, Kelley, and others.

Secondly, variates can be entered into the matrix for standardization in columns, for a population of attributes, so that for each person the following holds:—

$$\Sigma\, x_A^2 = \Sigma\, x_B^2 = \cdots = \Sigma\, x_N^2 = 1 \ . \qquad (2)$$

(where scores are summed in each column, from ${}_a x$ to ${}_n x$). Correlations are calculated between columns, for persons or whole aspects of persons as variables, and factorized by way of inverted factor theorems, that is, obverses of the theorems used in system (1).

Thirdly, variates which have been standardized as at (1) can be entered into the matrix, and thereupon *re-standardized* by way of system (2). That is, using $y$ to stand for variates that have already been standardized in rows, the same variates can be given the following standardization per column:—

$$\Sigma\, y_A^2 = \Sigma\, y_B^2 = \cdots = \Sigma\, y_N^2 = 1 \qquad (3)$$

(where the summation is per column, from ${}_a y$ to ${}_n y$). Correlations are for columns, and these, as in system (2), are factorized by way of inverted factor theorems.

Fourthly, scores already standardized as at (2) can be entered into the matrix and used for re-standardization by way of system (1). Re-standardization is effected with respect to the population of persons, for traits now considered separately as variables, the crude variates for which were standard scores with respect to a column. Using $z$ to stand for variates already standardized in columns, the following represents this system:—

$$\Sigma\, {}_a z^2 = \Sigma\, {}_b z^2 = \cdots = \Sigma\, {}_n z^2 = 1 \qquad (4)$$

(where the summation is now per row, from $z_A$ to $z_N$). Correlations are between rows, and are factorized by way of the usual Spearman or other factor theorems in their initial form, as in the case of system (1).

Systems (1) and (4) deal with separate* attributes as variables; (2) and (3) with persons. Factor measurements for an individual are regression estimates by way of systems (1) and (4); they are

*The word does not imply that the attributes are distinct, unique, isolated or unitary, in the sense of not having correlation with any others.

dependent upon the statistics of a population of persons. Those for systems (2) and (3) are correlation coefficients (loadings, or saturation coefficients). The original variates might be identical for all four systems of standardization, in which case the factor systems issuing from each will not be independent, although each has its own point of interest for psychometry. But the variates can also be quite independent for systems (1) and (2). Systems (3) and (4), on the other hand are not independent of (1) and (2) respectively.*

Systems (1) and (4) are what I called $r$-technique elsewhere, since they deal with the correlations ($r$) between tests or separate traits etc., and analysis is by way of direct factor theorems. Systems (2) and (3) have been called $Q$-technique, their concern being with correlations ($Q$) between *persons* or whole aspects of persons, analyzed by inverted factor theorems.

### *An Example.*

The following example will illustrate the above systems, and supply at the same time a standard of reference by which to judge past and future investigations involving factor analysis.

Suppose that we make 100 measurements ($a_1$ to $a_{100}$) for each of a number of persons $A$ to $N$. Let the measurements all relate to physical features of the body, height ($a_1$), arm-length ($a_2$), leg-length ($a_3$), trunk diameter ($a_4$), neck-circumference ($a_5$), finger length ($a_6$), and so forth. For the purposes of a controlled study, let the persons be of one age, and all comparable in point of nourishment and freedom from obesity etc. We can say a great deal about the factor systems for such measurements, purely on a priori grounds.†

First let the physical measurements be standardized in *rows* of the matrix of data (system (1)). That is, the measurements for height ($a_1$) are reduced to standard scores, for the population of $N$ persons, and similarly for each row separately. Such rows can be correlated, and it is obvious that the correlations must often be positive, and that a factor or factors will emerge for some of the variables, because the person with the greatest height will also tend to have the longest arm-length etc. The interpretation to be given to such factors will be discussed later.

But the crude measurements can also be standardized directly in *columns* (system (2)). All the 100 measurements are added, per per-

*The matrices can be built up in other ways: thus individuals might be measured for their Spearman, $g$, $p$, $o$, $w$, $f$, $c$, etc. in standard terms, and then correlated amongst themselves with these factor estimates as population.

†It seems worth while as an example to pursue an investigation for such measurements. I hope to publish such data in due course.

son, the mean and standard deviation for each person is calculated, and his crude scores reduced to standard ones. Such measurements can be correlated,* for persons as variables and the 100 physical attributes as population. The measurements are all considered entirely with respect to a person, without any quantitative reference to any other person. Different units of length can be employed for each person without making the slightest difference to the correlations. But it is obvious that all persons will correlate positively with each other (unless they are grossly deformed); indeed, the correlations will often be nearly perfect. It may be held at once that a general factor will be found for such data, whatever other orthogonal or other common factors may emerge. Again, however, I shall discuss what meaning to attach to such factors later.

If the measurements are *first* standardized as in system (1), and then correlated as in system (2), we have an example of system (3). All the attributes are now given equal weight, so to speak, a nose being no longer than a leg in standard terms. So far as any general factor of system (1) is concerned, this system (3) will show no counterpart to it: but any factors that emerge should not be independent of the subsidiary ones coming from system (1). System (3), however, can contribute information not already supplied by the previous systems; and it may often be convenient to use because its measurements for a person are factor saturations and not regression estimates.

An example of system (4) is supplied if the crude measurements are first standardized as for system (2), and *then* correlated as in system (1). If there is only one general factor (plus specific factors) in system (2), then no factors should emerge from system (4). Otherwise factors will usually appear, and will correspond in some way with any subsidiary common factors of system (2). Again, however, it is well to realize that it might sometimes be highly pertinent and fitting to measure attributes primarily relative to an individual as in system (2), and thereupon seek to factorize the attributes as variables with respect to a population of persons.

Thus four correlation matrices accrue in the above example. The first is for physical attributes as variables, for a population of persons; the second is for physical persons as variables,† for a popula-

*I purposely begin by calling attention to the standardization, which is, of course, automatically effected in the usual product-moment correlation coefficient $r = \frac{\Sigma x y}{n \sigma_x \sigma_y}$.

†As nearly whole as the 100 measurements allow: a large population of such measurements would allow a still closer approximation to the whole physical person.

tion of attributes, measured relative to each person himself;* the third is also for persons as variables, but the attributes have first been reduced so that all are *statistically* alike; the fourth is for physical attributes as variables, for a population of persons, but the attributes have all been gauged relative to each individual separately. It will not be denied that the matrices usually would contain significant correlations, and that factors would be discovered upon submitting these to factor analysis. It may also be granted that these factors, in the above example, with certain exceptions,† would be reflections of each other; that is, factors in system (1) would have a counterpart to some extent in each of the other systems (with the exceptions just mentioned).

Nevertheless very different methodological approaches are involved.

*Two Methodological Approaches.*

System (2) in the above example begins from the standpoint of the separate physical person. It neglects each person's *absolute* measurements, his absolute height etc. relative either to a population of persons or to an absolute physical standard (the metre or yard). Otherwise it can embrace in its population every physical measurement of a person, relative to himself.

It begins from the standpoint of the *whole* body-proportions of the person, and it can embrace all possible relations of the physical person in this respect. In a sense it can engulf the "total situation", for it takes all the physical measurements into account: but it always neglects the absolute features of all these physical measurements.

I shall show in due course that system (2) can in the same way embrace the relational aspects of a person in cognitive and orectic spheres no less than in this one of physical measurements. If we examine a person's moods, for instance, every perceivable mood-condition can enter into the population to be analyzed by way of system (2): but again we would in no way be concerned with the absolute quantities of any of the moods.

System (4) follows logically from (2), beginning from the standpoint of the proportions of body-parts relative to each individual himself. Each body-measurement is judged or gauged purely with respect

*Different units of measurement could be used for each person, so long as the same unit is used for all the measurements of the one person. The units could be inches for one person, centimetres for another, feet for another etc.

†Such as, that if only *one* factor covered the matrix for system (1), no broad factor could appear in system (3) in the above example.

to the individual *of which it is part.* In a sense the emphasis is still on the 'total' person, the "total situation".

System (1) begins with absolute measurements, in the sense of measurement in terms of a yard or metre. But it reduces all such measurements to standard terms, and the supposed absoluteness of the variates is merely a matter of relativity again — each person's height in standard terms is a measure made in terms of and relative to the heights of a population of persons, and similarly for each separate attribute.

It is true that system (1) could encompass the 'total' person in so far as all possible attributes (height etc.) are used as variables. But it completely neglects the relativity of such attributes, relative to each person, and it perpetrates a puzzling reduction of all the separate attributes to statistical terms of equal significance, irrespective of the relative significance of the attribute in the person. One cannot but have the suspicion that in the above example the system is dealing with severed heads, arms, noses, legs etc., piled into separate heaps, and that it will be impossible to find our way back to the whole person again.

I have already mentioned that although system (1) appears to begin with absolute variates, it does so only in a sense 'relative to a population of persons'. The system can certainly tell us if, and how, the various attributes vary proportionately in a population of persons. But it can tell us little or nothing about the whole body-build of any individual person. It supplies information of a *general* kind. There would seem to be only one set of conditions under which system (1) can give us information about the whole individual as such, namely, when all the variables are approximations to one and the same aspect or feature of individuals. This is the case, for instance, when mental tests are variables. The variates for each mental test can be legitimately reduced to standard terms; that the variables vary proportionately can be determined; but the information so supplied is pertinent to the *whole* individual, as Line* has shown. System (1) can supply undistorted information about mental test variates, because each represents, approximately, the same whole cognitive person in operation; each is an approximate measurement of *process,* measured in units of time. But unless some such common significance is attached to each variable, or *may* be so attached to each, system (1) cannot do other than distort the original facts, *so far as the individual*

*Line, W. *loc. cit.*

*is concerned,** as it undoubtedly does in the above physical example, and as it does, I shall later show, in work with temperament and character traits.

It is interesting to draw a comparison between systems (1) and (3), and (2) and (4), in terms of gestalt and the so-called atomistic theories in psychology. Gestalt theory scarcely needs a definition if it is identified with that of the Wertheimer-Koffka-Köhler school of thought. In particular this school rebels against the viewpoint that a whole is a sum of parts; the whole, rather, has primacy over any parts.†

Returning to our example of the physical attributes of persons, system (1) begins from the 'atomistic' standpoint. It begins with parts, and its factors also represent only parts of the individual. Thus, as is mentioned later, one factor may be specific to trunk measurements, and another to head measurements etc. Even the factors already known in system (1), such as *g*, *p*, *o*, *w*, etc., are also parts in the same sense.

It has hitherto been a criticism of system (1) that, having taken the person to pieces and measured his *g*, *p*, *o*, etc. separately, it could not proceed to put the person together again. The same criticism would apply, no less, to the physical variables or any factors arising from them. Now by applying system (3) to such part measures or factors, i.e. using the part measures as variates for a person, and thereupon correlating persons, we can certainly put these measures together again, with respect to each person. But having done so, system (3) by no means builds up the whole person again. Its factors can only be distorted, unreal, or potential, with respect to any individual, because it assumes the parts to be equally weighted for the individual at the outset — all the parts are given equal significance relative to the individual, when in fact they have not such equality. Beginning with parts in system (1), then, cannot lead to a *whole* person by way of system (3).

System (2), on the other hand does not begin with such equally weighted parts — instead, each is given its proper significance *relative to the whole of which it is part, and which governs these parts.* System (2) here seems to correspond with the gestalt standpoint in psychology; system (1) and (3) correspond no less with the atomistic standpoint. System (2) encompasses the "total situation"; system (1) does not really do so. System (1) begins by distorting the original

*So far as the population of persons is concerned it perpetrates no such distortion.

†Peterman, B. *The Gestalt Theory.* (Kegan Paul, London, 1932). (I refer to the "principle of the primacy of the whole over its parts".)

parts relative to the individual; system (2) effects no such distortion.

Nevertheless, as I have indicated, system (1) can always be used to supply information about the way in which any variables, for part or 'whole' aspects of a person, depend upon one another, i.e. it will show whether the variables are related, in the sense of varying proportionately in a population of populations, and with respect to that population. In this way, however, the system supplies information of interest to *general,* and not to *individual,* psychology, except when 'whole' aspects of persons are involved. As I have indicated, when the variables are for 'whole' aspects of persons, as is approximately the case for cognitive tests as variables, then system (1) not only supplies information of value to *general,* but also of essential interest to *individual* psychology. If, with Line, we consider the measurements supplied by mental tests (cognitive abilities, or functions or processes in general) to be for the 'whole' person in action, so that they have gestalt implications at the outset, then system (1) can supply information which in no way is a distortion of the original data—except in so far as any experimental abstraction from reality is artificial.

There can be no doubt that any data can be examined by either system (1) or (2). But there are circumstances under which it is most pertinent and proper to use either the one or the other, for otherwise the original data will be distorted.

*Further Observations.*

At risk of being tedious I add some further observations about the above example for physical attributes, and about the four standardization systems in general.

Any factor issuing from system (2) can have its nature completely estimated.* A factor so described is an estimate of the most typical (hypothetical) person for the given relational system of attributes or measurements. The factor is represented by a population of *measurements* (in standard terms) to which all persons of the factor are approximating in some degree, and the factor saturation of a person is an index of how closely he approximates to this 'typical' person.† The factor still refers to a 'total' person, except that it neglects the 'absolute' nature of the attribute, i.e. the relation of each to a

*See Stephenson, W. *A new application of correlation to averages.* Brit. J. Educ. Psych. VI. 1936. p. 43.

†I shall indicate in due course that factors in system (2) need not be analyzed orthogonally, and that thereby the " 'total' situation" is retained.

population of persons. This omission from the 'whole' person* is not so grievous a matter in system (2), as is the neglect of the relational features of the attributes in system (1).

Factors in system (1) are an indication that the separate variables entering into it vary proportionately in the population of persons. The person with the longest legs will tend also to have the longest arms, the widest chest, and the biggest feet etc. The essential nature of a factor in the latter case is represented by a population of *persons*, each person in which has his own constant factor amount ($g$, in standard terms), which tends to be the same in all the variables entering into the factor.† But this factor amount, which is estimated from a regression equation,‡ cannot be said to give us any description of the individual person. In the first place it is more than likely that any factor in system (1) will only be for *some* and not for *all* the physical measurements — one factor may be specifically connected with the trunk measurements, and another with head measurements, etc. This indicates at once that the factors are not encompassing the 'total' physical person. Secondly, the factor estimate for a *person* in system (1) only has a meaning relative to all the other persons involved; the estimate is relative to the population of persons.

System (3) concerns itself with a population of physical measurements, the same in name as for system (2). But the actual variates are all in standard terms — a nose is on the average no longer than a leg in standard terms, and system (3) begins with such measurements. A factor will no doubt emerge. *But its nature can scarcely be that of any 'real' physical person, because we have begun by mixing up the original measurements, so to speak, with those of other people.* We could therefore scarcely hope to return from the *general* (represented by the factor in system (3)) to the *particular* (the individual physical person).

System (4), on the other hand, does not perpetrate such an admixture in its original variates. Its factors again are an indication that the physical features entering into a factor vary proportionately, just as for system (1).

System (1), of course, is that used exclusively in the past in factor analysis. It supplies the basis for scientific work on empirically

*The word 'whole' must be taken here to mean relational together with 'absolute' aspects of the physical measurements.

†This is merely a literal description of what a theorem such as the Spearman two-factor (a common $g$ and specific $s$ factors) means.

‡In the Spearman two-factor theorem the amount is:

$$g = r_{ag} \cdot m \pm .6745\sqrt{1 - r_{ag}^2}\,.$$

observed or objectively determined *individual differences.** Although it begins with the premisses of individual differences, it only contemplates measuring any individual for those differences which enter into a factor. When a factor appears, such as Spearman *g*, there is some guarantee that all persons are alike in respect of certain individual differences. In this respect the system is primarily of interest to General rather than to Individual psychology.

System (2) does not begin from the standpoint of individual differences of any kind, but with empirical discoveries of a qualitative kind. Individual differences are in no way essential to it. But it *gives rise* to a kind of individual differences which has never before been attended to by psychometrists, namely, individual differences *in type.*† So far as system (2) is concerned, however, all persons might be of a type i.e. approximating to the same "typical" person, and they could also be identical in degree of factor saturation for the type. It just happens that in fact they are not. That different types emerge, and different degrees of each, is a matter for objective determination. Preliminary results show, as we might expect, that for many populations of traits or other data, all persons are *not* of a type.‡ The *distributions* of degrees of type for a population of persons is also a matter for determination from the observed facts: for eidetic type studies, for instance, the distribution of factor saturations may be markedly skewed. Only the facts can tell.

### *Conclusion.*

I cannot flatter myself that I have made everything in the above pages unequivocable, or even understandable. The main contributions, however, are clear enough.

1. The matrix of persons (row) and attributes (column) used by factorists can be employed in more ways than the one used almost exclusively in past factor analysis and psychometry generally. Four ways have been described and illustrated.

2. Two of these four (1, 3) have atomistic foundations unless they deal with *process* measurements. The other two (2, 4) have relativistic and gestalt implications.

3. The criterion for use of system (1) would seem to be that the variables are equally significant for the individual; when they are

*The words refer here, and always in the sequel, to differences with respect to a population of persons.

†The word 'type' is here and throughout this paper used in a special sense, so far as I know never before used. It is the subject of a paper to follow this one.

‡See examples in my various introductory papers.

not, then system (2) is the appropriate system to begin one's investigation with, from the standpoint of the individual.

4. There can be no doubt, I think, that system (2) deals most faithfully with the original data in the case of the physical measurements, from the standpoint of the individual.

5. System (4) supplies a means for investigating whether and how separate* attributes vary proportionately. It would seem to be the gestalt counterpart of system (1), but system (1) is more obviously applicable to studies of this *general* kind, i.e. the varying proportionalities of attributes for a population of persons.

I would conclude by reiterating that there is no question in the above of suggesting that systems (2) and (4) should *replace* systems (1) and (3) in psychometry. It is obvious that all four can be used, each in its appropriate place. But system (2) can deal with 'total' persons, whereas system (1) only does so for process measurement.

* * * * * * * * * *

*Note added July 10th, 1936.*

In the above paper I have been concerned with certain particular and general applications of the four factor systems. Only a brief definition of each system was given, a full account being reserved for the main work upon which I have been engaged for some time, which is to be published shortly. The above account seemed to be sufficient for the present purpose, which was to draw attention to other possibilities in factor analysis, and to other applications, than those widely known to most workers. But the brevity of the definitions, and certain special features of the example by which I have illustrated them, may lead to misunderstandings which the fuller treatment will obviate. Meanwhile, the following further notes may forestall misinterpretations, at least.

The important systems are (1) and (2), the other two being of minor immediate interest. I have clearly stated that systems (1) and (2) can be quite independent, but it will be better to state explicitly that I regard the two as, by very definition, statistically independent of one another in general. It will be helpful, in this connection, to distinguish between two different forms that system (2), or (1), can take.

System (2) in general is not to be regarded as the direct obverse or mere transpose of data already analyzable by way of system (1); nor is it the case that these two systems are merely two complementary ways of analyzing one and the same original matrix of data, the results being no less complementary or deducable one from the other. Such complementary conditions, and the possibility of direct analysis either in rows or in columns, is only possible under the special case of universality of *unit*. One must first define what units are involved in either the rows or the columns. It is not sufficient to put scores into a matrix and then say without more ado that they can be analyzed either by rows or by columns. Of course they *could* be; but unless something is told us about the units involved, such two-way analysis will usually be a futile and useless proceeding. It is precisely the purpose of systems (3) and (4) to ensure, under the conditions of non-universality of unit in either rows or columns respectively, a general reduction to comparable units, so that analysis would then be sensible and possibly useful in general. In the very special case, however, when one and the same unit

*The word again refers only to the mode of thinking about the attributes. Systems (1) and (4) proceed to show whether the attributes vary proportionately, and when any two so vary neither can be regarded as isolated or distinct.

of measurement is involved in *both* columns and rows by definition (as could be the case in the example of the physical attributes described above, if an *inch* is used throughout as the unit of measurement, for all persons and all attributes), direct analysis by either rows or columns is sensible and permissible. In this case the same original data, without any reduction of any kind, can be analyzed either by system (1) or by system (2).

In the general case, however, the data is by very definition distinct for analysis by way of system (1) or (2) respectively. In system (1) it is not in the least essential to have one and the same unit for all *attributes* or *tests*; it is merely essential that the unit for any one attribute should be one and the same for the whole population of persons. In system (2), likewise, it is not essential to have one and the same unit for all *persons*; it is merely essential that the unit for any one person should be the same for the whole population of attributes involved for that person. The units for system (1) moreover, are generally by definition or fact different from any of those involved in system (2), so far as we know, or need know.

It so happens that I consider this non-universality of unit, in either rows or columns respectively, to be a matter of the very first importance. In the study of personality, for instance, matters of the greatest possible interest depend upon this fact, that analysis can proceed without a universal unit of measurement.

The burden of the above paper, however, amongst others, was that of showing that even when we begin with special data of the kind used in the example, for the 100 physical attributes, and analyze it by both system (1) and system (2), the two different standardizations effected (when one correlates the rows, or the columns), effect a radical alteration in the original data, and therefore make the two subsequent factor analyses really incomparable. The two are only dependent in so far as both use the same crude data; but the different standardizations effect profound changes in this crude data. As I have already said, analysis by way of system (2) could proceed even if *different* units are used for each *person* respectively; but in that case direct obverse analysis of that same data by way of system (1) would be a futile and meaningless proceeding. The same is true, of course, if the original data is for unique or different units per *attribute* (but the same for all persons, per attribute); in such a case subsequent analysis of the same data as it stands, by way of system (2), would be futile and meaningless. But even if one and the same unit is used throughout, for all persons and all attributes, the respective standardizations to which the data is submitted (in effect) when either rows or columns are correlated, radically change the original data. In effect, therefore, one is not dealing with the same data in the two systems, for factor analysis only concerns standardized material, and not crude data. It would have been better to have stated this explicitly or more emphatically.

In the above paper I wrote (largely so as to allow me to press on with the essential argument) that the various factors arising in the different systems might, to some extent, be "reflections" of one another. Had space permitted, or intention inclined me to it, I should have elaborated this considerably. Actually I go on to show that the factors will in fact be very different in the various systems, both as to essential nature and as to interpretation, and this is generally the case. They are really only superficial "reflections", as matters of interpretation.

I had intended expanding on matters of the above kind in my fuller study of system (2), for there are still other interesting considerations involved. In general however, in system (2) I am concerned to adumbrate a system which will be the very poles apart from system (1). System (1) involves a population of *persons*, but there need only be *one* attribute for investigation; system (2) involves a population of attributes, but there might only be a *single* person involved. (I shall supply examples in due course of factor analysis, performed upon myself alone as variable.) I therefore demanded quantification which could be confined to a single person, uniquely if need be, and system (2) supplies a means for dealing with such quantifications. Similarly, in system (1) there is an unlimited population of persons from which to draw one's samples, and sampling error theory is pertinent. But in system (2) there may be a restricted population of attributes, so few that one's 'sample' constitutes the whole universe

of these attributes; I have therefore to contemplate other than sampling error theory.

The statistical differences between systems (1) and (2), then, are radical and fundamental. Usually, by very definition we begin with data the units for which are expressly defined with respect to system (1), or with respect to system (2), and there is no necessary connection at all between them. The psychological applications of these systems are no less profoundly different.

In the above paper I have tried to indicate very briefly some of these psychological applications. System (1) has its counterpart in the 'atomistic' standpoint in psychology, whereas system (2) encompasses a gestalt standpoint. I shall, of course, have much more to say about this in future work, but sufficient has been said, I think, to point towards this important conclusion. I may be pardoned, however, if I elaborate a little on two matters touched upon very briefly in the above paper.

Factor analysis by way of system (1), as I shall show in detail elsewhere, is concerned primarily with *tendencies*, i.e. with the way in which attributes vary one with another in a population of persons. It begins from the standpoint of observed *individual differences* (relative to a population of persons), and arrives at information about abilities, processes, unitary traits, primary abilities, or in general, fundamental tendencies. Each such tendency or process is an explanation for the observed facts, namely, that such and such variables vary proportionately and uniquely. The factors are by definition universal. And it seems to be a fact that there is only a limited number of such fundamental tendencies in the human being: Spearman found only five or so; Thurstone specifies seven; the Thorndike Unitary Traits Committee hoped to find from 1 to 20. The implication is that these few fundamental tendencies account for, or explain, or are the cause of, all human conduct. Right through history there has been a search for such tendencies, or else a talk about them, from the classical humours of Hippocrates or Galen, to the classical Introversion-Extraversion of Jung, and now the same kind of thought is involved in the search for unitary traits. System (1) is the method for investigating such fundamental tendencies. I do not as yet deny its efficacy in this respect, but I draw attention to the fact that the system is limited to such investigations.

System (2) accomplishes something very different indeed. It can supply no direct information about fundamental tendencies of the kind just referred to. Instead it deals with whole personalities, so-called 'irrelevancies', in general with 'total situations' in what I take to be the gestalt meaning of these words. It deals with attributes relative to each individual, and to himself alone, and each attribute has its full and proper context (in so far as this is reasonably possible in any analysis) in the whole of which it is part. This is excellently illustrated, I think, by application of system (2) to the 100 physical measurements described above. Like all analysis, system (2) has to parcel out its applications. It can only deal with whole *aspects* of persons, with the physical whole, the mood-condition whole, the cognitive whole, and so forth. All measurement is relative to the individual, and to himself alone. It has no necessary concern *at all* with individual differences of the system (1) *a priori* or *a posteriori* kind — instead it gives rise to its own kind of individual differences, those of type and type saturation or loading. Its factors are not universal necessarily; they *may* be, but only as a matter of *a posteriori* objective determinations.

In the physical example I postulated 100 attributes. To cover the physical whole more completely, or more determinately, this population might be increased to 1000, or to 10,000 if need be. But even 20 of these attributes might define the physical whole of all persons sufficiently approximately. It would have to be shown that whatever population is selected from amongst all the possible physical attributes of a person, the factor systems only altered in determinacy, and not in essential nature. In such a case we could legitimately say that we are dealing with a real physical whole, a 'total situation'. It seems too obvious to mention that a whole situation can never be encompassed if by this we mean, as Thurstone apparently means, the infinite universe of all possible attributes and all possible persons. In system (2), as in system (1), we can only deal with a finite population of either persons or attributes. Even so, system (1) is fundamentally 'atomistic' (fortunately, for it is its special merit that it allows an approach to

the study of fundamental tendencies), whereas system (2) deals with 'total situations' in the above gestalt sense. In system (1) each attribute is stripped from its total context, each is standardized in terms of a population of persons (which destroys all possibility of returning to the individual wholes of the person). The 'total situation' can *never* be realized in system (1), even if the whole universe of attributes are used as variables. It *may* be retained intact in system (2), with more or less determinacy, for as few as ten, or even less, attributes.

I have said that there are possibly only a few fundamental tendencies in the human being, and therefore only a few unitary traits are born out of system (1) analysis. But there are possibly millions of *types*, each a common factor, (common that is to several or many persons, but not necessarily to all), in system (2) analysis. Type psychology comes into its own by way of system (2). Almost everything that we factorists have denied about type psychology has been beside the point: types exist in great number, and one person can be of one type and not at all of another; types are *not* extremes of otherwise normal distributions, if by type we mean the interpretation given to a common factor in system (2) analysis.

PSYCHOMETRIKA—VOL. 1, NO. 3
SEPTEMBER, 1936

# THE APPROXIMATION OF ONE MATRIX BY ANOTHER OF LOWER RANK

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The mathematical problem of approximating one matrix by another of lower rank is closely related to the fundamental postulate of factor-theory. When formulated as a least-squares problem, the normal equations cannot be immediately written down, since the elements of the approximate matrix are not independent of one another. The solution of the problem is simplified by first expressing the matrices in a canonic form. It is found that the problem always has a solution which is usually unique. Several conclusions can be drawn from the form of this solution.

A hypothetical interpretation of the canonic components of a score matrix is discussed.

## *Introduction*

If $N$ individuals are each subjected to $n$ tests, it is a fundamental postulate of factor theory that the resulting $n \times N$ score matrix $a$ can be adequately approximated by another matrix $\beta$ whose rank $r$ is less than the smaller of $n$ or $N$. Closely associated to this postulate is the purely mathematical problem of finding that matrix $\beta$ of rank $r$ which most closely approximates a given matrix $a$ of higher rank $R$. It will be shown that if the least-squares criterion of approximation be adopted, this problem has a general solution which is relatively simple in a theoretical sense, though the amount of numerical work involved in applications may be prohibitive. Certain conclusions can be drawn from the theoretical solution which may be of importance in practical work.

To formulate the problem precisely, it is convenient to define the "scalar product" of two $n \times N$ matrices as the following numerical function of their elements:

$$(a, \beta) = \sum_{i=1}^{n} \sum_{j=1}^{N} a_{ij} \beta_{ij} \qquad (1)$$

where $a_{ij}$ is the $ij$-element of the matrix $a$. This function has all the properties of the scalar product of vectors:

$$(a, \beta) = (\beta, a) \quad ; \qquad (2)$$

$$(\alpha \pm \beta, \gamma) = (\alpha, \gamma) \pm (\beta, \gamma) \ ; \tag{3}$$

$$(x\alpha, \beta) = x(\alpha, \beta) \quad \text{if } x \text{ is a number;} \tag{4}$$

$$(\alpha, \alpha) > 0 \quad \text{if } \alpha \neq 0 \ . \tag{5}$$

The positive number $l$, defined by $l^2 = (\alpha, \alpha)$ may be called the length of the matrix $\alpha$; the number $l^2$ was called the span (Spannung) of $\alpha$ by Frobenius.* The length of the matrix $\alpha - \beta$ may be called the distance from $\alpha$ to $\beta$.

The problem can now be formulated in a definite manner: to find that matrix $\beta$ of rank $r$, such that there is no other matrix of rank $r$ whose distance from $\alpha$ is less than the distance from $\beta$ to $\alpha$. This amounts to requiring a least-squares solution of the approximation problem, every element of the given matrix being given equal weight.

*Some preliminary theorems and remarks.*

To simplify the following discussion, let $a, b, u, \cdots$ denote $n \times n$ matrices, $A, B, U, \cdots$ denote $N \times N$ matrices, and $\alpha, \beta, \cdots$ denote $n \times N$ matrices; the special case $n = N$ is not excluded. The $N \times n$ matrix which is obtained by writing the columns of $\alpha$ as rows is the transpose of $\alpha$ and will be denoted by $\alpha'$. The products $a\alpha$, $\alpha A$, $\alpha B'$, $\alpha' a, \cdots$ are defined in the usual way. It is then seen that the following equations are correct:

$$(\alpha, \beta) = (\alpha', \beta') \ ; \tag{6}$$

$$(a\alpha, \beta) = (\alpha, a'\beta) = (\alpha, \beta a') \ ; \tag{7}$$

$$(\alpha A, \beta) = (\alpha, \beta A') = (A, \alpha'\beta) \ . \tag{8}$$

From Eq. (7) and (8) it follows that if $u$ and $U$ are orthogonal matrices ($u\,u' = u'\,u = 1_n$, $U\,U' = U'\,U = 1_N$), then

$$(u\,\alpha\,U', u\,\beta\,U') = (\alpha, \beta) \ . \tag{9}$$

Another useful proposition is the following: if $(\alpha, b) = 0$ for all symmetric (skew-symmetric) matrices, $b$, then $\alpha$ is skew-symmetric (symmetric).

The solution of the problem is much simplified by an appeal to two theorems which are generalizations of well-known theorems on square matrices.† They will not be proven here.

*Quoted from MacDuffee, "Theory of Matrices", Ergebn. d. Mathem., v. 2, No. 5 p. 80 (1933).

†Courant and Hilbert, "Methoden der mathematischen Physik" Berlin, 1924; pp. 9 et seq., p. 25. MacDuffee, p. 78.

*Theorem I. For any real matrix $a$, two orthogonal matrices $u$ and $U$ can be found so that $\lambda = uaU'$ is a real diagonal matrix with no negative elements.*

A diagonal matrix $\lambda$ (square or rectangular) is one for which $\lambda_{ij} = 0$ unless $i = j$. If a diagonal matrix is rectangular, then there will be some rows or columns which consist entirely of zeros. For the following, this remark is of some importance, as will be seen. The equation of the theorem may also be written

$$a = u' \lambda U \tag{10}$$

whose right side may be called the canonic resolution of $a$. If $n < N$, $\lambda$ will have $N$—$n$ columns of zeros and $a$ is seen to depend only on the first $n$ rows of $U$. If $u$, $\lambda$ and the first $n$ rows of $U$ are given, $a$ is determined.

Let $v$ be the diagonal $n \times n$ matrix which consists of the first $n$ columns of $\lambda$, and $\omega$ the $n \times N$ matrix composed of the first $n$ rows of $U$; then these remarks can be summarized by the equation

$$a = u' v \omega \tag{10.1}$$

where $\omega \omega' = 1_n$, but $\omega' \omega \neq 1_N$. For numerical work, Eq. (10.1) is preferable to Eq. (10); for formal manipulation, Eq. (10) is more convenient.

The numerical evaluation of $u$ and $v$ (or $\lambda$) can be accomplished from the consideration of the matrix $\alpha = a a'$ alone. This matrix is closely related to the matrix of correlation coefficients of the tests. It is seen that

$$\alpha = a a' = u' \lambda \lambda' u = u' v^2 u \ ,$$

and since $\lambda \lambda' = v^2$ is a diagonal matrix, it follows that $u$ is one of the orthogonal matrices which transform the correlational matrix to diagonal form. The rows of $u$ are unit vectors along the principal axes of $\alpha$ and the squares of the diagonal elements of $v$ (or $\lambda$) are the characteristic values of $\alpha$; this shows that the latter can never be negative numbers, a result which can also be obtained more directly.* The methods for determining the principal axes and characteristic values of a symmetric matrix are also known,† so that these remarks may be considered as indicating the method for calculating $u$ and $\lambda$. If none of the characteristic values of $\alpha$ is zero (this will presumably be the case in the overwhelming proportion of actual calculations)

*Courant-Hilbert, p. 20.
†Courant-Hilbert, pp. 13, 16.

the matrix $v$ will have a reciprocal, and $\omega$ can be obtained by solving Eq. (10.1):

$$\omega = v^{-1} u a \ .$$

The numerical values of the elements in the remaining rows of the matrix $U$ will not be needed, but could be found if necessary. For simplicity of manipulation, it is convenient to proceed as though this has been done.

The diagonal elements of $\lambda$ were called the "canonical multipliers" of $a$ by Sylvester.* The multipliers and characteristic values of a correlational matrix are identical; in the case of a symmetric matrix, there may be a difference in sign; for a general square matrix, there is no simple relation between the two; for a rectangular matrix, the characteristic values are not defined.

The correlational matrices, Sylvester called the "false squares" of $a$. In the foregoing (and in the usual treatment of factor theory) only the matrix $a = a\,a'$ has been considered. However, the matrix $A = a'a$ is related to the correlation coefficients of the individuals in the same manner as $a$ is related to the correlation coefficients of the tests. There is complete mathematical symmetry between the two correlation matrices.

To every multiplier, there is associated a row of $u$ and a row of $U$; this complex of $n + N + 1$ numbers may be called a canonic component of $a$.

*Theorem II. If $a\,\beta'$ and $\beta'\,a$ are both symmetric matrices, then and only then can two orthogonal matrices $u$ and $U$ be found such that $\lambda = u\,a\,U'$ and $\mu = u\,\beta\,U'$ are both real diagonal matrices.*

Either one (but in general, not both) of the diagonal matrices may be further restricted to have no negative elements. This theorem is a generalization of the theorem that the principal axes of two symmetric matrices coincide if and only if $ab = ba$.

*Solution of the problem*

The distance of $\beta$ from $a$ is given by $x$, where

$$x^2 = (a, a) - 2(a, \beta) + (\beta, \beta) \ ; \qquad (11)$$

$x$ is a function of all the elements of $\beta$, and these are to be determined so that its value is a minimum. The elements of $\beta$ are not all independent, however, because of the requirement that its rank be less than the number of its rows or columns. Theorem I makes it possible

*Messeng. Math., *19* p. 45 (1889).

to eliminate some of the interdependence: suppose $\beta$ to have been resolved into canonic form:

$$\beta = u' \mu U \tag{12}$$

with $\mu$ diagonal, and $u$ and $U$ orthogonal matrices. Then the rank of $\beta$ will be $r$ if and only if $\mu$ has this rank i.e., if just $r$ of the diagonal elements of $\mu$ are different from zero; the non-vanishing elements of $\mu$ will be independent. However, the elements of $u$ or $U$ will not be independent, since these matrices must be orthogonal. It is not necessary to express these matrices in terms of independent parameters because of the following proposition:* if $u$ is any orthogonal matrix and the independent variables that determine it are given any infinitesimal increments, the resulting increment of $u$ is

$$\delta u = u s \ , \tag{13}$$

where $s$ is a skew-symmetric matrix whose elements are infinitesimal, but otherwise arbitrary.

The Eq. (11) becomes, because of Eq. (12) and (9),

$$x^2 = (\alpha, \alpha) - 2(\alpha, u' \mu U) + (\mu, \mu) \ . \tag{14}$$

Since $x$ is to be a minimum, it follows that $\delta x^2 = 0$ when $u$ is given the increment $\delta u$ (Eq. (13)).
Hence

$$0 = (\alpha, -s u' \mu U) = -(\alpha, s \beta) = -(\alpha \beta', s) \ . \tag{15}$$

Since $s$ is an arbitrary skew-symmetric matrix, it follows that $\alpha \beta'$ must be symmetric. Discussing the increment of $U$ in the same manner, it will be found that $\beta' \alpha$ must also be symmetric, and hence, by Theorem II, the orthogonal matrices can be found so that Eq. (12) and

$$\alpha = u' \lambda U \tag{12.1}$$

(with $\lambda$ the diagonal matrix of the multipliers of $\alpha$) are both valid. Then Eq. (11) becomes

$$\begin{aligned} x^2 &= (\lambda - \mu, \lambda - \mu) \\ &= \textstyle\sum_i (\lambda_i - \mu_i)^2 \end{aligned} \tag{11.1}$$

$\lambda_i$ and $\mu_i$ being the diagonal elements of the corresponding matrices.

It remains to determine the matrix $\mu$ so that this expression has its minimum value, subject to the condition that just $r$ of the $\mu_i$ shall

*Courant-Hilbert, p. 27.

be different from zero. It may be supposed that $\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \cdots$ ; then an obvious solution of the problem is

$$\mu_i = \lambda_i \ , \quad i \leqq r \ ;$$
$$= 0 \ , \quad i > r \ . \tag{17}$$

This is also the only solution unless $\lambda_r = \lambda_{r+1}$ .

The procedure of finding $\beta$ may be summarized: (1.) express $a$ in the canonic form of Eq. (12.1) ; (2.) replace all but $r$ of the diagonal elements of $\lambda$ by zeros, beginning with the smallest and continuing in order of increasing magnitude. (3.) The resulting matrix is $\mu$, and $\beta$ is given by Eq. (12). The solution is unique unless $\lambda_r = \lambda_{r+1}$ .

The minimum value of $x^2$ is

$$x^2 = \sum_{i=r+1}^{R} \lambda_i^2$$

($R$ being the rank of $a$ and the multipliers being numbered as before). This leads readily to the following conclusion: if $l$ is the length of $a$, the smallest upper bound for $x$ is $l(1—r/R)^{1/2}$. This can be utilized in estimating the significance of an approximation to $a$ obtained by some other method than the present one.

Another important conclusion is that if $\beta$ is the best approximation (of rank $r$) to $a$, then $b = \beta\,\beta'$ ($B = \beta'\,\beta$) will also be the best approximation (of rank $r$) to $a = a\,a'$ ($A = a'\,a$). Thus, if the best approximation to the score matrix is found, then the correlational matrices calculated from it will automatically be the best approximations to the correlational matrices calculated from the original score matrix. This is not surprising, but requires proof; the proof is readily supplied from the foregoing results.

### *Concerning a hypothetical interpretation of the canonic components of $\beta$*

It is reasonable to inquire if, when $a$ is a score matrix, those of its canonic components that enter into $\beta$ may be interpreted as the independent factors that determine the differences in the performance of various individuals. In order to discuss this question let $p = 1, \cdots n$ number the tests, $i = 1, \cdots N$ number the individuals, and $\varrho$ or $\sigma = 1, \cdots r$ number the components of $\beta$ that have non-vanishing multipliers. Then the component $\varrho$ consists of the $n + N + 1$ numbers

$$u_{\varrho p} \ , \quad \lambda_\varrho > 0 \ , \quad U_{\varrho i}$$

which satisfy the equations

$$\sum_p u_{\rho p} u_{\sigma p} = \delta_{\rho\sigma} \; ; \quad \sum_i U_{\rho i} U_{\sigma i} = \delta_{\rho\sigma} \tag{18}$$

$$\beta_{pi} = \sum_\rho u_{\rho p} \lambda_\rho U_{\rho i} \; . \tag{19}$$

These equations suggest that if these numbers have any empirical significance, then $N^{1/2} U_{\rho i}$ will be the standard score of individual $i$ for the ability to exercise the factor $\varrho$, and $n^{1/2} u_{\rho p}$ the standard score of the test $p$ for its demand for the exercise of the factor. The appearance of the multipliers in Eq. (19) does not correspond to the accepted postulates of factor theory, and their interpretation is thus not immediate.

To clarify this point, one may inquire after those empirical circumstances, which if they are actually realized, will lead one to accept the foregoing interpretation. One such set of circumstances would be the following: suppose that two sets of individuals, $N_{\text{I}}$ and $N_{\text{II}}$ in number, have been tested by the same battery of $n$ tests. The scores of the $N_{\text{I}}$ individuals can then be arranged in an $n \times N_{\text{I}}$ matrix $a_{\text{I}}$, those of the $N_{\text{II}}$ individuals into an $n \times N_{\text{II}}$ matrix $a_{\text{II}}$; or, the scores of all $N_{\text{I}} + N_{\text{II}}$ individuals can be arranged into an $n \times (N_{\text{I}} + N_{\text{II}})$ matrix $a$ obtained by adjoining $a_{\text{I}}$ and $a_{\text{II}}$. First consider $a_{\text{I}}$ and $a_{\text{II}}$; if these can be approximated by matrices $\beta_{\text{I}}$ and $\beta_{\text{II}}$ having the same $u$-matrices* (within reasonable limits of error), then the foregoing interpretation would be appropriate. In this case the $u$-matrix obtained by approximating $a$ would also be the same as the others. The multipliers $\lambda_{\rho,\text{I}}$, $\lambda_{\rho,\text{II}}$ and $\lambda_\rho$, and of course also the $U_{\rho i,\text{I}}$, $U_{\rho i,\text{II}}$ and $U_{\rho i}$ obtained from the three score-matrices would not be identical. In order to discuss their interrelations, let the index $i$ now run from 1 to $N_{\text{I}} + N_{\text{II}}$, the first $N_{\text{I}}$ values designating individuals of the first set, etc. Then it can be shown that

$$\lambda_\rho{}^2 = \lambda_{\rho,\text{I}}{}^2 + \lambda_{\rho,\text{II}}{}^2 \tag{20}$$

and

$$\begin{aligned} \lambda_\rho U_{\rho i} &= \lambda_{\rho,\text{I}} U_{\rho i,\text{I}} \text{, if } i \leqq N_{\text{I}} \; ; \\ &= \lambda_{\rho,\text{II}} U_{\rho i,\text{II}} \text{, if } i > N_{\text{I}} \; . \end{aligned} \tag{21}$$

These are precisely the relations that would be expected according to the suggested interpretation if $\lambda_{\rho,\text{I}}{}^2/N_{\text{I}}$ and $\lambda_{\rho,\text{II}}{}^2/N_{\text{II}}$ were the variances of the two sets of individuals. This is not a tenable interpretation, however, because of the symmetric manner in which the

*The sufficient condition for this equality is that $a_{\text{I}}\, a_{\text{II}}$ be a symmetric matrix, where $a_{\text{I}} = \alpha_{\text{I}}\alpha_{\text{I}}'$, etc. This criterion can be applied even when $u_{\text{I}}$ and $u_{\text{II}}$ are not known.

individuals and the tests enter into the calculations. It may, however, be supposed that $\lambda_{\rho,1}{}^2/nN_1$ is the product of the variance of the set of individuals with the variance of the set of tests. It should be noted that it has been implicitly assumed that the scores are not measured from the averages of the different sets of individuals, but from a fiducial zero which is the same for all sets.

PSYCHOMETRIKA—VOL. 1, NO. 3
SEPTEMBER, 1936

# A FURTHER SIMPLIFICATION OF THE MULTIPLE AND PARTIAL CORRELATION PROCESS

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**Formulas are derived for simplified computation of partial and multiple correlation coefficients, and generalized to $n$ variables. Time required for computation is compared with other methods.**

## 1. *Introduction*

Psychometricians and others who work with the multiple and partial correlation techniques and with regression equations will find that the methods presented in this article will save an appreciable amount of time and labor. The greater the number of variables, the more apparent will be the saving effected by these new formulas.

Several years ago the writer developed and published fundamental formulas for the Doolittle method in terms of zero-order correlation,* but his failure to discern relationships within the formulas to known statistical concepts other than $r$'s and $\beta$'s prevented him from expanding the formulas beyond four variables at that time.

The formulas were:

### Three Variables

$$\beta_{02.1} = (r_{02} - r_{01}\, r_{12}) / (1 - r_{12}^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [1a]$$

$$\beta_{01.2} = r_{01} - \beta_{02.1}\, r_{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [1b]$$

$$R_{0.12} = (\beta_{01.2}\, r_{01} + \beta_{02.1}\, r_{02})^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [2]$$

### Four Variables

$$\beta_{03.12} = \frac{r_{03} - r_{01}\, r_{13} - \dfrac{r_{02} - r_{01}\, r_{12}}{1 - r_{12}^2}\,(r_{23} - r_{12}\, r_{13})}{1 - r_{13}^2 - \dfrac{r_{23} - r_{12}\, r_{13}}{1 - r_{12}^2}\,(r_{23} - r_{12}\, r_{13})} \quad . \quad . \quad . \quad . \quad [3a]$$

$$\beta_{02.13} = \frac{r_{02} - r_{01}\, r_{12}}{1 - r_{12}^2} - \beta_{03.12}\,\frac{r_{23} - r_{12}\, r_{13}}{1 - r_{12}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad [3b]$$

*Griffin, Harold D. "Fundamental Formulas for the Doolittle Method Using Zero-Order Correlation Coefficients." *Annals of Mathematical Statistics II* (May, 1931), pp. 150-153.

$$\beta_{01.23} = r_{01} - (\beta_{02.13}\, r_{12} + \beta_{03.12}\, r_{13}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [3c]$$

$$R_{0.123} = (\beta_{01.23}\, r_{01} + \beta_{02.13}\, r_{02} + \beta_{03.12}\, r_{03})^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [4]$$

The writer was encouraged to seek concepts to express various recurring parts of these formulas by Professor Walter J. Saupe of the University of Missouri and Professor George W. Snedecor of Iowa State College. As a result, formulas for problems in three to eight variables were published in 1933* in which two new concepts were introduced—$\alpha$ (Greek, alpha) for the numerator and $\omega$ (Greek, omega) for the denominator of the partial regression coefficient, $\beta$ (Greek, beta).

The present article contains a still further simplification of the formulas, with elimination of the concept $\omega$ by substitution of the more familiar concept $k^2$, the coefficient of non-determination.† A further extension enables the new formulas to be used not only for rapid and precise calculation of the multiple correlation constants but for partial correlation problems as well.

The concepts and their symbols which we shall employ in these new formulas are as follows:

$N$ = the number of cases or observations
$n$ = the number of variables
$\nu$ = (Greek, nu) the number of independent variables = $n - 1$
$M$ = the mean, arithmetic mean
$\sigma$ = standard deviation
$r$ = correlation coefficient
$a$ = incomplete regression coefficient, alpha coefficient
$k^2$ = non-determination coefficient
$k$ = alienation coefficient
$\beta$ = partial regression coefficient, beta coefficient.
$R^2$ = multiple determination coefficient
$R$ = multiple correlation coefficient
$b$ = regression coefficient

*Griffin, Harold D. "Simplified Schemas for Multiple Linear Correlation." *Journal of Experimental Education I* (March, 1933), pp. 239-254.

†The following changes will adjust the schemas in the article mentioned in the preceding footnote to the new notation:

$$\begin{array}{llll}
\omega_{23} = k^2_{23} & \omega_{24.3} = k^2_{4.23} & \omega_{25.34} = k^2_{5.234} & \omega_{26.345} = k^2_{6.2345} \\
\omega_{24} = k^2_{24} & \omega_{25.3} = k^2_{5.23} & \omega_{26.34} = k^2_{6.234} & \omega_{27.345} = k^2_{7.2345} \\
\omega_{25} = k^2_{25} & \omega_{26.3} = k^2_{6.23} & \omega_{27.34} = k^2_{7.234} & \omega_{28.345} = k^2_{8.2345} \\
\omega_{26} = k^2_{26} & \omega_{27.3} = k^2_{7.23} & \omega_{28.34} = k^2_{8.234} & \\
\omega_{27} = k^2_{27} & \omega_{28.3} = k^2_{8.23} & & \omega_{27.3456} = k^2_{7.23456} \\
\omega_{28} = k^2_{28} & & & \omega_{28.3456} = k^2_{8.23456}
\end{array}$$

$C$ = constant in the regression equation calculated from the means
$X$ = a raw value or score
$\overline{X}$ = predicted raw value or score
$S$ = standard error of estimate
$\sigma_\beta$ = standard deviation of a beta coefficient
$r_{01}$ = zero-order correlation coefficient
$r_{01.2}$ = partial correlation coefficient

### *2. Three-Variable Problems*

We present below the set-up for a three-variable problem in which the object is to obtain the minimum essentials for prediction—the beta coefficients, the regression equation, the multiple correlation coefficient, and the standard error of estimate.

Required Zero-Order Data

$$M_0, M_1, M_2 \; . \quad \sigma_0, \sigma_1, \sigma_2 \; . \quad r_{01}, r_{02}, r_{12} \; .$$

First-Order Calculations

$$a_{02.1} = r_{02} - r_{01}\, r_{12} \qquad k^2_{01} = 1 - r^2_{01}$$
$$k^2_{12} = 1 - r^2_{12} \; .$$

Final Beta Coefficients

$$\beta_{02.1} = a_{02.1}/k^2_{12}$$

$$\beta_{01.2} = r_{01} - \beta_{02.1}\, r_{12}$$

Check: $r_{02} = \beta_{01.2}\, r_{12} + \beta_{02.1}$ .

Multiple Correlation Coefficient

$$k^2_{0.12} = k^2_{01} - \beta_{02.1}\, a_{02.1}$$

$$R_{0.12} = (1 - k^2_{0.12})^{\frac{1}{2}}$$

Alternative: $R_{0.12} = (\beta_{01.2}\, r_{01} + \beta_{02.1}\, r_{02})^{\frac{1}{2}}$ .

Regression Equation

$$b_{01.2} = \beta_{01.2}\, \sigma_0/\sigma_1$$

$$b_{02.1} = \beta_{02.1}\, \sigma_0/\sigma_2$$

$$C_{0.12} = M_0 - (M_1\, b_{01.2} + M_2\, b_{02.1})$$

$$\overline{X}_0 = b_{01.2}\, X_1 + b_{02.1}\, X_2 + C_{0.12} \; .$$

Standard Error of Estimate

$$S_{0.12} = \sigma_0\, k_{0.12} \; .$$

In a three-variable problem each of the final béta coefficients, $\beta_{01.2}$ and $\beta_{02.1}$, will have the same standard deviation. The three-variable formula for the standard deviation of a beta coefficient is

$$\sigma_{\beta 01.2} \text{ and } \sigma_{\beta 02.1} = (k^2_{0.12}/k^2_{12}N)^{\frac{1}{2}} .$$

When the partial correlation coefficients are desired in a three-variable problem, the following additional steps are necessary:
Additional First-Order Calculation

$$k^2_{02} = 1 - r^2_{02} .$$

Partial Correlation Coefficients

$$r_{01.2} = (1 - k^2_{0.12}/k^2_{02})^{\frac{1}{2}}$$

$$r_{02.1} = (1 - k^2_{0.12}/k^2_{01})^{\frac{1}{2}} .$$

The partial correlation coefficient always takes the sign of its corresponding beta coefficient. Therefore, if $\beta_{01.2}$ is positive, $r_{01.2}$ will also be positive.

### 3. *Four-Variable Problems*

The four-variable set-up for prediction is as follows:
Required Zero-Order Data

$$M_0, M_1, M_2, M_3 . \quad \sigma_0, \sigma_1, \sigma_2, \sigma_3 .$$

$$r_{01}, r_{02}, r_{03}, r_{12}, r_{13}, r_{23} .$$

First Order Calculations

$$a_{02.1} = r_{02} - r_{01} r_{12} \qquad k^2_{01} = 1 - r^2_{01}$$

$$a_{03.1} = r_{03} - r_{01} r_{13} \qquad k^2_{12} = 1 - r^2_{12}$$

$$a_{23.1} = r_{23} - r_{12} r_{13} \qquad k^2_{13} = 1 - r^2_{13}$$

$$\beta_{02.1} = a_{02.1}/k^2_{12}$$

$$\beta_{32.1} = a_{23.1}/k^2_{12} .$$

Second-Order Calculations

$$a_{03.12} = a_{03.1} - \beta_{02.1} a_{23.1}$$

$$k^2_{0.12} = k^2_{01} - \beta_{02.1} a_{02.1}$$

$$k^2_{3.12} = k^2_{13} - \beta_{32.1} a_{23.1} .$$

Final Beta Coefficients

$$\beta_{03.12} = a_{03.12}/k^2_{3.1}$$

$$\beta_{02.13} = \beta_{02.1} - \beta_{03.12}\,\beta_{32.1}$$

$$\beta_{01.23} = r_{01} - (\beta_{02.13}\,r_{12} + \beta_{03.12}\,r_{13})$$

Check: $r_{02} = \beta_{01.23}\,r_{12} + \beta_{02.13} + \beta_{03.12}\,r_{23}$ .

Multiple Correlation Coefficient

$$k^2_{0.123} = k^2_{0.12} - \beta_{03.12}\,a_{03.12}$$

$$R_{0.123} = (1 - k^2_{0.123})^{\frac{1}{2}} .$$

Regression Equation

$$b_{01.23} = \beta_{01.23}\,\sigma_0/\sigma_1$$

$$b_{02.13} = \beta_{02.13}\,\sigma_0/\sigma_2$$

$$b_{03.12} = \beta_{03.12}\,\sigma_0/\sigma_3$$

$$C_{0.123} = M_0 - (M_1\,b_{01.23} + M_2\,b_{02.13} + M_3\,b_{03.12})$$

$$\overline{X}_0 = b_{01.23}\,X_1 + b_{02.13}\,X_2 + b_{03.12}\,X_3 + C_{0.123} .$$

Standard Error of Estimate

$$S_{0.123} = \sigma_0\,k_{0.123} .$$

When the standard deviations of the beta coefficients are wanted in a four-variable problem, it will be necessary to add the following steps to the foregoing procedure:

Additional First-Order Calculations

$$a_{13.2} = r_{13} - r_{12}\,r_{23} \qquad k^2_{23} = 1 - r^2_{23}$$

$$\beta_{13.2} = a_{13.2}/k^2_{23}$$

$$\beta_{23.1} = a_{23.1}/k^2_{13} .$$

Additional Second-Order Calculations

$$k^2_{1.23} = k^2_{12} - \beta_{13.2}\,a_{13.2}$$

$$k^2_{2.13} = k^2_{12} - \beta_{23.1}\,a_{23.1} .$$

Standard Deviations of the Beta Coefficients

$$\sigma_{\beta_{01.23}} = (k^2_{0.123}/k^2_{1.23}N)^{\frac{1}{2}}$$

$$\sigma_{\beta_{02.13}} = (k^2_{0.123}/k^2_{2.13}N)^{\frac{1}{2}}$$

$$\sigma_{\beta_{03.12}} = (k^2_{0.123}/k^2_{3.12}N)^{\frac{1}{2}} .$$

When the partial correlation coefficients are required in a four-variable problem, it is necessary to add the following formulas to the basic set-up:

Additional First-Order Calculations

$$a_{03.2} = r_{03} - r_{02}\, r_{23} \qquad k^2_{02} = 1 - r^2_{02}$$

$$k^2_{23} = 1 - r^2_{23}$$

$$\beta_{03.1} = a_{03.1}/k^2_{13}$$

$$\beta_{03.2} = a_{03.2}/k^2_{23} \ .$$

Additional Second-Order Calculations

$$k^2_{0.13} = k^2_{01} - \beta_{03.1}\, a_{03.1}$$

$$k^2_{0.23} = k^2_{02} - \beta_{03.2}\, a_{03.2} \ .$$

Partial Correlation Coefficients

$$r_{01.23} = (1 - k^2_{0.123}/k^2_{0.23})^{\frac{1}{2}}$$

$$r_{02.13} = (1 - k^2_{0.123}/k^2_{0.13})^{\frac{1}{2}}$$

$$r_{03.12} = (1 - k^2_{0.123}/k^2_{0.12})^{\frac{1}{2}} \ .$$

The partial correlation coefficient always takes the sign of its corresponding beta coefficient. Thus, if $\beta_{01.23}$ is negative, $r_{01.23}$ will also be negative.

### 4. *Generalized Formulas*

The concepts and formulas which we have employed in this article are generalized below. The subscripts 1, 2, 3, etc., refer to the sequence of the subscripts in the formulas, and $n$ refers to the number of variables. Other symbols will be explained as they occur in the formulas.

(First order incomplete regression coefficient, alpha coefficient —three variables. $a_{12.3} = a_{21.3}$)

$$a_{12.3} = r_{12} - r_{13}\, r_{23} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [5a]$$

([$n$—2]$^{\text{th}}$ order alpha coefficient—$n$>3 variables.

$a_{12.34\cdots n} = a_{21.34\cdots n}$)

$$a_{12.34\cdots n} = a_{12.34\cdots(n-1)} - \beta_{1n.34\cdots(n-1)}\, a_{n2.34\cdots(n-1)} \quad . \quad [5b]$$

(First order non-determination coefficient—three variables)

$$k^2_{12} = 1 - r^2_{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [6a]$$

([$n$—1]$^{th}$ order non-determination coefficient—$n>3$ variables)

$$k^2_{1.23\cdots n} = k^2_{1.23\cdots(n-1)} - \beta_{1n.23\cdots(n-1)}\,\alpha_{1n.23\cdots(n-1)} \quad . \quad [6b]$$

([$n$—2]$^{th}$ order intermediate partial regression coefficient, beta coefficient—$n$ variables. This is the formula used in obtaining intermediate beta coefficients; the formulas used for the final beta coefficients are given under Formula 8. $\beta_{12.34\cdots n} \neq \beta_{21.34\cdots n}$)

$$\beta_{12.34\cdots n} = \alpha_{12.34\cdots n}/k^2_{2.34\cdots n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [7]$$

(Final beta coefficients—$n$ variables. In these generalized formulas for the final beta coefficients, the subscript 0 indicates the dependent variable or criterion and the subscript $\nu$ [Greek nu] indicates the number of independent variables. $\beta_{0\nu.12\cdots(\nu-1)} \neq \beta_{\nu 0.12\cdots(\nu-1)}$)

$$\beta_{0\nu.12\cdots(\nu-1)} = \alpha_{0\nu.12\cdots(\nu-1)}/k^2_{\nu.12\cdots(\nu-1)} \quad . \quad . \quad . \quad . \quad . \quad [8a]$$

$$\beta_{0(\nu-1).12\cdots\nu} = \beta_{0(\nu-1).12\cdots(\nu-2)} -$$
$$\beta_{0\nu.12\cdots(\nu-1)}\,\beta_{\nu(\nu-1).12\cdots(\nu-2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [8b]$$

$$\beta_{0(\nu-2).12\cdots\nu} = \beta_{0(\nu-2).12\cdots(\nu-3)} -$$
$$(\beta_{0(\nu-1).12\cdots\nu}\,\beta_{(\nu-1)(\nu-2).12\cdots(\nu-3)} +$$
$$\beta_{0\nu.12\cdots(\nu-1)}\,\beta_{\nu(\nu-2).12\cdots(\nu-3)}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [8c]$$

. . . . . .

$$\beta_{03.12\cdots\nu} = \beta_{03.12} - (\beta_{04.12\cdots\nu}\,\beta_{43.12} +$$
$$\beta_{05.12\cdots\nu}\,\beta_{53.12} + \cdots + \beta_{0\nu.12\cdots(\nu-1)}\,\beta_{\nu 3.12}) \quad . \quad . \quad . \quad [8x]$$

$$\beta_{02.13\cdots\nu} = \beta_{02.1} - (\beta_{03.12\cdots\nu}\,\beta_{32.1} +$$
$$\beta_{04.12\cdots\nu}\,\beta_{42.1} + \cdots + \beta_{0\nu.12\cdots(\nu-1)}\,\beta_{\nu 2.1}) \quad . \quad . \quad . \quad . \quad [8y]$$

$$\beta_{01.23\cdots\nu} = r_{01} - (\beta_{02.13\cdots\nu}\,r_{12} +$$
$$\beta_{03.12\cdots\nu}\,r_{13} + \cdots + \beta_{0\nu.12\cdots(\nu-1)}\,r_{1\nu}) \quad . \quad . \quad . \quad . \quad . \quad [8z]$$

([$n$—1]$^{th}$ order multiple correlation coefficients—$n$ variables)

$$R_{0.12\cdots\nu} = (1 - k^2_{0.12\cdots\nu})^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [9a]$$

([$n$—2]$^{th}$ order regression coefficient—$n$ variables. $b_{01.23\cdots\nu} \neq b_{10.23\cdots\nu}$)

$$b_{01.23\cdots\nu} = \beta_{01.23\cdots\nu}\,\sigma_0/\sigma_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [10a]$$

$$b_{02.13\cdots\nu} = \beta_{02.13\cdots\nu}\,\sigma_0/\sigma_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [10b]$$

etc.

(Constant in regression equation for raw scores—$n$ variables)

$$C = M_0 - (b_{01.23\cdots\nu}\,M_1 + b_{02.13\cdots\nu}\,M_2 + \cdots$$
$$\cdots + b_{0\nu.12\cdots(\nu-1)}\,M_\nu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [11]$$

(Regression equation in terms of raw scores—$n$ variables)

$$\overline{X}_0 = b_{01.23\ldots\nu} X_1 + b_{02.13\ldots\nu} X_2 + \cdots$$
$$\cdots + b_{0\nu.12\ldots(\nu-1)} X_\nu + C \quad [12]$$

(Standard error of estimate—$n$ variables)

$$S_{0.12\ldots\nu} = \sigma_0 k_{0.12\ldots\nu} \quad [13]$$

(Standard deviation of the beta coefficients—$n$ variables)

$$\sigma_{\beta_{01.23\ldots\nu}} = (k^2_{0.12\ldots\nu}/k^2_{1.23\ldots\nu} N)^{\frac{1}{2}} \quad [14a]$$

$$\sigma_{\beta_{02.13\ldots\nu}} = (k^2_{0.12\ldots\nu}/k^2_{2.13\ldots\nu} N)^{\frac{1}{2}} \quad [14b]$$

etc.

(Partial correlation coefficient—$n$ variables. In these generalized formulas for the partial correlation coefficient, the subscript 0 indicates the dependent variable and the subscript $\nu$ indicates the number of independent variables. The partial correlation coefficient always takes the sign of its corresponding beta coefficient. Although $\beta_{01.23} \neq \beta_{10.23}$, we will find that $r_{01.23} = r_{10.23}$; this is because $r_{01.23} = (\beta_{01.23}\,\beta_{10.23})^{\frac{1}{2}}$ )

$$r_{01.234\ldots\nu} = (1 - k^2_{0.123\ldots\nu}/k^2_{0.234\ldots\nu})^{\frac{1}{2}} \quad [15a]$$

$$r_{02.134\ldots\nu} = (1 - k^2_{0.123\ldots\nu}/k^2_{0.134\ldots\nu})^{\frac{1}{2}} \quad [15b]$$

. . . . . .

$$r_{0(\nu-1).123\ldots\nu} = (1 - k^2_{0.123\ldots\nu}/k^2_{0.123\ldots(\nu-2)\nu})^{\frac{1}{2}} \quad [15y]$$

$$r_{0\nu.123\ldots(\nu-1)} = (1 - k^2_{0.123\ldots\nu}/k^2_{0.123\ldots(\nu-2)(\nu-1)})^{\frac{1}{2}} \quad [15z]$$

In addition to these fundamental formulas there are two possible checks on operations. Formula 16 should always be applied when final beta coefficients are computed.

(Check on the calculation of the final beta coefficients—$n$ variables)

$$r_{02} = \beta_{01.23\ldots\nu} r_{12} + \beta_{02.13\ldots\nu} + \beta_{03.12\ldots\nu} r_{23} +$$
$$\beta_{04.12\ldots\nu} r_{24} + \cdots + \beta_{0\nu.12\ldots(\nu-1)} r_{2\nu} \quad [16]$$

(Alternative calculation of the multiple correlation coefficient $n$ variables)

$$R_{0.12\ldots\nu} = (\beta_{01.23\ldots\nu} r_{01} + \beta_{02.13\ldots\nu} r_{02} + \cdots$$
$$\cdots + \beta_{0\nu.12\ldots(\nu-1)} r_{0\nu})^{\frac{1}{2}} \quad [9b]$$

### 5. *Comparisons*

These generalized formulas should enable the psychometrician to arrange schemas with which to solve his own problems. The greatest saving will, of course, be effected in those problems for which these new formulas were originally intended—the calculation of the beta coefficients, the multiple correlation coefficient, the standard error of estimate, and the construction of the regression equation. In Table I we have compared the number of arithmetic processes required to find $\beta_{01.2345}$, $\beta_{02.1345}$, $\beta_{03.1245}$, $\beta_{04.1235}$, and $\beta_{05.1234}$ in a six-variable problem by Moore's determinental method* with the number required by the writer's new formulas. The saving effected by the latter is readily apparent.

The new formulas, however, are also economical of time and effort in solving partial correlation problems. For example, if we wish merely to obtain the partial correlation coefficients without the other constants, we shall find that for the same number of significant fig-

TABLE I

COMPARISON OF THE NUMBER OF OPERATIONS REQUIRED BY DETERMINANTS (MOORE'S METHOD) AND THE NEW FORMULAS IN SOLVING FOR

$\beta_{01.2345}$, $\beta_{02.1345}$, $\beta_{03.1245}$ $\beta_{04.1235}$, AND $\beta_{05.1234}$

IN A SIX-VARIABLE PROBLEM

| | Determinants | New Formulas |
|---|---|---|
| Additions | 24 | 6 |
| Subtractions | 56 | 30 |
| Multiplications | 122 | 36 |
| Divisions | 15 | 10 |
| Tables | 7 | 4 |

*Moore, Thomas Verner. *Multiple Correlation and the Correlation between General Factors.* Catholic University of America Studies in Psychology and Psychiatry, vol. III, no. 1.

ures retained in intermediate operations the new formulas will obtain the solutions with less calculations and with greater accuracy than will the older formulas of the type:

$$r_{01.23\cdots\nu} = \frac{r_{01.23\cdots(\nu-1)} - r_{0\nu.23\cdots(\nu-1)}\, r_{1\nu.23\cdots(\nu-1)}}{(1 - r^2_{0\nu.23\cdots(\nu-1)})^{\frac{1}{2}}\,(1 - r^2_{1\nu.23\cdots(\nu-1)})^{\frac{1}{2}}}$$

Table II contains a comparison of the number of operations required by the old and new methods in finding the partial correlation coefficients, $r_{01.234}$, $r_{02.134}$, $r_{03.124}$ and $r_{04.123}$, in a five-variable problem. Again the efficiency of these new formulas is demonstrated.

TABLE II

COMPARISON OF THE NUMBER OF OPERATIONS REQUIRED BY THE OLD AND THE NEW FORMULAS IN SOLVING FOR

$r_{01.234}$, $r_{02.134}$, $r_{03.124}$, and $r_{04.123}$

IN A FIVE-VARIABLE PROBLEM

| | Old Formulas (Yule's) | New Formulas (Griffin's) |
|---|---|---|
| Additions | 0 | 0 |
| Subtractions | 42 | 39 |
| Multiplications | 44 | 28 |
| Divisions | 22 | 16 |
| Squares | 20 | 7 |
| Square Roots | 20 | 4 |